THE LIMITS OF SCIENCE

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THE LIMITS OF SCIENCE

Outline of Logic and of the Methodology of the Exact Sciences

BY
THE LATE

LEON CHWISTEK, PH.D.

INTRODUCTION AND APPENDIX
By Helen Charlotte Brodie, M.A.

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" They say miracles are past ; and we have our philosophical persons, to make modern and familiar, things supernatural and causeless."

WILLIAM SHAKESPEARE, *All's Well that Ends Well*, Act ii, Scene 3.

Translated from the Polish
by Helen Charlotte Brodie, M.A.
and
Arthur P. Coleman, Ph.D.
Lecturer in East European Languages,
Columbia University

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AUTHOR'S PREFACE TO ENGLISH EDITION

The English edition of *Granice Nauki* is essentially different from the original text.

Chapter VII is completely changed. In Chapters VIII and IX important additions have been made.

I am greatly indebted to Miss Brodie, Mgr. Herzberg, and Dr. Hetper for important critical remarks.

LEON CHWISTEK.

Lwów,

TRANSLATORS' PREFACE

Significant contributions have been made by contemporary Polish thinkers in the fields of logic, philosophy of science, and the analysis of the foundations of mathematics. They have been the initiators and leaders of contemporary thought on many important issues involved in metalogic and linguistics. Relatively little of their work is available in any international language and hence the writings of these theoreticians have remained comparatively unfamiliar to students in these fields in other countries of Europe and in America.

The Limits of Science, by Dr. Leon Chwistek, was first published in 1935 under the title *Granice Nauki.* The present edition has been revised and supplemented by the author. The translators are indeed grateful to Dr. Chwistek for his whole-hearted co-operation both in revising the work and carefully checking the manuscript.

Taking the views of Bertrand Russell, Henri Poincaré, and David Hilbert as his point of departure, Dr. Chwistek goes on to develop rational semantics, which he contends can be successfully applied in solving the problems which arise in connection with philosophy, science, social theory, and art. *The Limits of Science* is the culmination of Dr. Chwistek's thought with regard to the application of rational semantics to logic, the philosophy of mathematics, and the foundation problems of the physical sciences.

It is perhaps unnecessary to point out, especially to the Polish reader, that the translators have directed their efforts toward a free translation rather than a word-for-word rendering of the text. It has proved more feasible to eliminate certain idiomatic expressions of the Polish language and allusions familiar only to the Polish reader, and to concentrate our efforts upon obtaining an adequate and coherent interpretation of the text. Wherever possible translations of quotations from works in foreign languages have been taken directly from the English translation of these works.

The translators wish to express their thanks to Professor Herbert W. Schneider who initially encouraged this project; Professors Haskell B. Curry, A. F. Bentley, and Rudolf Carnap, whose recognition of its value motivated its execution; Dr. J. Herzberg, of Lwów, who gave invaluable assistance in the tedious task of checking references ; Professor Horace L. Friess, who gave freely of his time in checking references, interpreting allusions, and discussing certain problems which arose in connection with the translation ; Dr. H. Theodric Westbrook and Dr. Ernest Moody, who willingly offered suggestions and criticisms in rendering quotations taken from the medieval Latin ; to Dr. Josef Maier, who verified translations of quotations taken from German authors, and to Miss Jean Macalister, of the Columbia University Library, who checked several obscure references.

The translators are deeply indebted to Professor Ernest Nagel without whose efforts the publication of this translation would have been impossible. He not only undertook to make the initial arrangements for publication, offered his advice with regard to the problems which arose in connection with the work, but checked the manuscript in its entirety, offering invaluable suggestions and criticisms with regard to terminology and interpretation.

In offering this translation of *The Limits of Science* to the philosophical public it is the hope of the translators that this initial translation of a logical text from Polish into English will not be the last, that an increase in the familiarity of Western thinkers with the works of Polish theoreticians written in their native tongue will follow and that a more adequate understanding and evaluation of their contributions will be obtained.

H. C. B.
A. P. C.

PREFACE TO INTRODUCTION AND APPENDIX

Chwistek's views on logic, and in particular those concerning semantics and metamathematics, were developed over a period of many years. However a study of his writings of the last four or five years reveals that, except for matters of detail, his views have attained their final form. For this reason it is important to indicate explicitly how the present text differs in form from the original edition. Chwistek himself points out that:

> "In Chapters IV–VI instead of the Greek letters α, β, . . . the letters *u*, *v*, *w*, . . . are employed. Otherwise there will be no conformity with the system of Chapter VII. Symbols such as **(0000)**, etc., have no individual meaning; they are not names at all. To have significant propositions we must assume that **(0000)** *is true*, or that it is a theorem."

In a series of letters written during the summer of 1939 Chwistek dealt specifically with the varieties of type to be used in setting up the manuscript and submitted certain general directions, which can be summarized as follows:

1. Italics are to be employed in the case of sentences of the symbolic language, and in the case of real and apparent variables (both logical and semantical) which are not starred expressions. They are also to be used to indicate phrases or sentences which are emphasized.
2. Bold face is to be employed in the case of the language of interpretation *and* the interpreted language. Constant expressions, logical operators, and variables (whether real or apparent), which are defined as starred expressions, are also to be printed in bold face.
3. All mathematical symbols, when not considered within the context of the system of semantics or metamathematics, are to be written in accordance with the usual mathematical conventions.

The application of these directions was left in all cases to the present writer. Unfortunately many questions of interpretation arose in this connection and Dr. Chwistek was unable to read the final draft of the manuscript in which they were resolved. While Professor Ernest Nagel aided immeasurably in dealing with them, the actual responsibility for the choice of type of all symbols must rest upon the writer.

It might be added here that in any case it would have been impossible to follow the typography of the original edition not only because of fundamental changes in Chwistek's position since its publication in 1935, but because of the inclusion in the text of a considerable amount of hitherto unpublished material.

In the Introduction and Appendix an attempt is made to develop a consistent interpretation of Chwistek's views, to eliminate all their "obscurities", and to give an adequate evaluation of them. It is therefore necessary to employ terminology current among other logicians as well as Chwistek's own phraseology. Consequently on occasion deviations from Chwistek's terminology and notation may be found on certain fundamental points.[1] Quotation marks, for example, are employed to indicate the name of an expression. Although Chwistek himself does not accept this convention, it readily permits the reader to discover exactly what Chwistek has in mind at a given point. While this and other reformulations employed in the Introduction and Appendix have been given only after a careful consideration of Chwistek's views in his own terms, they are essential if his position is to be understood and evaluated by other logicians. Since, however, Chwistek never saw these portions of the text it is impossible to decide whether he would be willing to accept the writer's interpretation of his views exactly as they stand. The reader can test its adequacy by an examination of the translation itself, where Chwistek's own symbolism and notation remain unchanged.

Finally the writer wishes to express her appreciation to Wellesley College, under whose auspices the Introduction and Appendix were completed during her term as Alice Freeman Palmer Fellow (1939–1940).

H. C. B.

[1] However in actual quotations from Chwistek's writings, all his conventions (past and present) are followed, except where specifically indicated. All translations from the Polish were made by the writer.

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INTRODUCTION

For almost three decades Dr. Leon Chwistek has been occupied with the problems of philosophy and logic. While it is generally recognized that he has made valuable contributions to logic by his critical analysis of its foundations, his own doctrines in philosophy and logic have been the subject of heated debate, particularly in his native land. *The Limits of Science* presents Chwistek's position on some of these controversial issues. While admittedly but an outline of the methodological and logical problems of the exact sciences, this work merits consideration as an attempt to approach these problems from the standpoint of a new logical science, semantics, and a new view concerning reality, the theory of plural reality. In the light of current philosophical discussion it is of interest to note that Chwistek was led to work out this new approach as a result of a prior analysis of language. His position can therefore be characterized as the reaction of a present-day nominalist to contemporary realistic and anti-rationalist doctrines.

Unfortunately, however, Chwistek uses a vocabulary and symbolic apparatus different from that of other philosophers and logicians. A number of writers have tried to restate his views in more familiar terminology. Consequently differences of opinion have arisen concerning the proper interpretation of his position as well as its validity. Chwistek has defended his position largely by attacking that of his opponents. But just as Chwistek's critics have never really tried to understand his views in *his* terms, he has never really tried to understand the viewpoint of his critics. An adequate evaluation of his point of view requires that some common basis of discussion be attained. It is the aim of this introduction to supply this lack and to contribute to such an evaluation. An attempt will therefore be made to place Chwistek's logical views in their proper philosophical setting. The influence of his logical theories upon more general philosophical considerations will also be examined.

I

The Method of Sound Reason

The Criteria of Sound Reason

It is a commonplace to state that philosophers and scientists both seek a coherent conception of the world and attempt to give an adequate analysis of experience. In view of this identity of aim it is not entirely surprising that their views have had an influence upon each other. Nevertheless the scientist and philosopher treat different aspects of experience. The physicist for example may be concerned with the conditions for the occurrence of electrodynamic phenomena, while the philosopher may be concerned with the generalized problems of knowledge.

It is Chwistek's contention, as a result of long preoccupation with philosophic and scientific problems, that certain weaknesses inherent in scientific procedure have given rise to many false philosophic doctrines. For example the inability of the Greek philosophers to remove the paradoxes discovered by Zeno (192 ff.)[1] gave rise to a philosophy of " pure being ". Thus in times of crisis in the history of science philosophers have been wont to advance doctrines in which exact thought is replaced by vision and phantasy. In this way Chwistek explains the widespread influence of Plato's thought. He interprets Hegel's views as arising from the confusion among eighteenth and nineteenth century mathematicians concerning the nature of infinitesimals.

Chwistek himself protests against any philosophic doctrine which is based upon " absolutes ", because they cannot be exemplified in or verified by experience. For this reason he objects to such concepts as " the perfect good " of Socrates, the " ideas " of Plato, the " absolute truth " of Hegel, and the " absolute knowledge " of Husserl. Philosophic doctrines, he maintains, are to be secured by the application of reason (i.e. sound reason) to experience. Only in this way is it possible to attain knowledge and add to the scope of our experience.[2]

Chwistek's analysis of sound reason has obviously been

[1] References to the text will be inserted in parenthesis.

[2] Chwistek also maintains that metaphysical elements must be eliminated from science. He objects, for example, to the introduction of entelechies into biology by Driesch (5). Nevertheless he insists that some method must be found to eliminate such scientific puzzles as the already mentioned paradoxes of Zeno. Once again Chwistek has recourse to the method of sound reason.

motivated by the situation in philosophy where philosophers, who present utterly incompatible views, each claim to have achieved knowledge. Chwistek himself believes that it is possible to discover certain fundamental truths, which though perhaps trivial, are not subject to variations in interpretation. These propositions, which are open neither to serious dispute nor alteration, he regards as the foundation of science and philosophy. Chwistek clearly formulates his view, when he defines sound reason as "the method for attaining truths not subject to intellectual revolution" (25).

Unfortunately, however, it is difficult to discover exactly what Chwistek means by sound reason. While he recognizes that it consists of a number of fundamental assumptions,[1] he asserts that "its criteria cannot be formulated in a pattern" (265), and that these criteria are variable (płynny).[2] He freely admits that "the exact bounds of their operation cannot be fixed" (265). The net result of his discussion of sound reason is therefore merely a statement of some of the well-known features of the reflective method, although these features do not characterize this method completely.

Chwistek has selected for consideration various criteria of the method of sound reason. But in the case of each of these criteria he recognizes its inadequacy as a defining characteristic of this method. Sound reason, for example, relies upon habits, but habits are subject to alteration. Again, sound reason works successfully only in the domain of familiar phenomena. Even the laws of thought, which are also advanced as positive criteria of this method, are subject to these limitations (29–30).[3] Nor can Occam's razor (43) guarantee reliable knowledge. As a rule of selection, which requires the acceptance of the simpler of two alternative explanations, it has a negative role; but even in this capacity this principle cannot be formulated precisely.

[1] Contrary to the procedure of certain present day philosophers, Chwistek admits the dependence of his views upon certain assumptions. Cf. e g. *Z.M C.*, p. 186, " . . . they " [the philosophers] " forget only too often that the demonstration of anything requires the acceptance of some supposition . . . the acceptance of suppositions is an arbitrary act and is . . . conditioned by a certain feeling of truth which, however, is undoubtedly subjective and cannot be forced upon any one as necessary."

[2] *W.R.*, p. 46.

[3] Chwistek points out the validity of the principle of contradiction with respect to definite questions which require definite answers. But he also points out the necessity of specifying certain supplementary and frequently artificial conditions in the case of propositions involving change (29–30). Cf. also *Z.S.*, p. 276.

Chwistek also characterizes sound reason as a method which involves criticism. But while he realizes the importance of being aware of the function of reason in science and philosophy he also recognizes the part played by the emotions, intuition, and background of the scientist and philosopher in the development of their views. For this reason the method of sound reason cannot be identified solely with criticism. For the exercise of sound reason requires not only criticism but "construction". Unfortunately Chwistek's usage of the latter term is not free from ambiguity. In his treatment of the natural sciences and the problem of reality he uses this term as a synonym for the synthesis of concepts. In the case of the deductive sciences he evidently has in mind the construction of systems.

Chwistek's consideration of these characteristics of the method of sound reason shows that none of them formulate the method adequately. Each of these characteristics must be regarded as referring only to a partial method, whose application in conjunction with other such partial methods constitutes an application of the method of sound reason. A criterion for the failure to use the method of sound reason in some particular analysis, according to Chwistek, is that one of these partial methods has not been employed. For example, he regards Hegel's doctrines as anti-rational [1] because they are incompatible with one of the fundamental principles of sound reason, the principle of contradiction (12–14). Many other citations might be offered in support of this interpretation of the method of sound reason.

[1] Chwistek uses the word "anti-rational" together with the terms "metaphysical", "idealistic", and "fictional" as derogatory epithets A doctrine is "anti-rational" if it is not obtained by the application of the method of sound reason. The terms "metaphysical", "idealistic", and "fictional" are used to refer to concepts which have no experiential base and consequently cannot be verified by reference to experience. For example, Chwistek regards Newton's absolute space as a metaphysical, idealistic, and fictional concept. This general position is familiar to the reader of contemporary positivistic literature. It should be noted, however, that Chwistek extends the usual list of terms of opprobrium far beyond its usual length.

It is of course possible to quarrel with Chwistek's terminology since he assigns new meanings to familiar philosophical terms. However his general intent is clear enough. It is therefore important to point out that Chwistek does not feel that problems which are usually called metaphysical are either meaningless or idealistic, when conceived as the study of the fundamental problems of existence. He does not for example hold that the problem of the relation between the soul and body is meaningless (cf. *W.R.*, pp. 39–40). He discusses the problem of free will at some length (*W.R.*, pp. 40–1, 54–5). This recognition of the possibility of metaphysics clearly distinguishes his views from those of many present-day positivists.

Language in the Light of Sound Reason

The formulation of the results of scientific and philosophical research in purely linguistic terms necessitates a language suitable for precise investigations, to be used in conformity with the principles of sound reason. Consequently Chwistek attempts to formulate some of the criteria of meaningful discourse. In the main he follows the British empirical tradition in identifying the meanings of a term with the ideas or images evoked by it. Accordingly, when "meaning" is so conceived, a term will vary considerably from individual to individual and from situation to situation.[1] It is therefore not difficult for Chwistek to show the falsity of the view that concepts have an absolute "real" meaning, which is the same for all individuals. Nor does Chwistek find any merit in the view, advanced by writers such as Husserl, that there are apriori laws for distinguishing the meaningful from the meaningless.

Chwistek then raises the question whether everyday language is an instrument suitable for scientific and philosophical purposes. His answer, which is in the negative, is based largely upon the theory of meaning which he proposed. Everyday language contains many abstractions which are treated as concrete objects. As Shestov says, "Just as things of the external world have a real existence for us, so the good has a real existence for Socrates" (27). Plato regards the soul as an object in everyday use (28). Such general concepts are subject to individual interpretation. Because the same term is used in different meanings in everyday language it is not difficult to construct contradictions in this language (40–2). Leonard Nelson, for example, has uncovered the following paradoxical[2] situation in epistemology (271)[3]:

> Epistemology is concerned with the problem whether or not objective knowledge is possible. To solve this problem it is assumed that there exists some criterion which can be applied in its solution. This criterion must obviously either be knowledge or not.
>
> If this criterion is knowledge it belongs to the domain whose validity is being examined and is therefore problematical.

[1] It is not Chwistek's intent to dispense with any of these meanings nor with any of the terms current in philosophical and scientific discourse. He requires only that the meanings of the terms used be clearly and carefully specified.

[2] A paradox or antinomy is a statement which can be shown to be both true and false on the basis of the same set of premises.

[3] Cf. *W.R.*, pp. 38–9.

Consequently the criterion to be used in solving the epistemological problem cannot itself be knowledge.

On the other hand if the criterion to be used is not knowledge, then it itself must be known, i.e. the criterion of knowledge must be employed.

While such difficulties may at first sight seem trivial they have far-reaching consequences for an adequate logic and philosophy of mathematics. Since it is possible to construct such contradictions in everyday language, this language is obviously not consistent and the rules which govern its construction and usage cannot guarantee that it will function correctly. It is not then a language which is suitable for scientific and philosophical purposes.

If, however, Nelson's epistemological paradox is examined more closely it will be noted that the paradox is obtained only by using two different senses of the word "knowledge" interchangeably. The word "knowledge" has been made to refer to itself. It has been suggested by some writers that paradoxes can be eliminated by postulating that a concept or statement cannot be used to refer to itself. This suggestion has been worked out in various ways, and the rules proposed for the attainment of this end are called theories of types. Chwistek has made a positive contribution to the theory of meaning [1] by suggesting a theory of types for *everyday language*, with the help of which such contradictions as the epistemological paradox [2] will be eliminated from this language. He has thus outlined a device for preventing the assigning of a single property to different types of entities.

The difficulties which Chwistek finds in everyday language have led him to adopt a position which he calls "nominalism". For example, his rejection of abstract ideas and universals (xxii, xxv) leads him to maintain that the scientist and philosopher

[1] Chwistek even goes so far as to recognize that the term "meaning" is not itself entirely unambiguous and he suggests the possibility of a hierarchy of propositions formed on the basis of different meanings of the word "meaning". Cf. *W.R.*, p. 85, and *Z.S.*, p. 330.

[2] On Chwistek's view the epistemological paradox involves the sentence, "The criterion for the epistemological problem is knowledge," and the sentence, "'The criterion for the epistemological problem is knowledge' is knowledge." In the first sentence the property of "being knowledge" is predicated of a noun. In the second sentence this property is predicated both of a noun and of a sentence. The property of "being knowledge" when predicated of a noun is of lower type than when predicated of a sentence. It has not the same meaning in both cases. If these facts are realized it becomes impossible to reason in the manner indicated by Nelson and the paradox in question cannot arise.

should be concerned only with concrete objects. While he therefore frowns upon the use of general concepts, individual words, and names play an important role in Chwistek's conception. But because of the ambiguity of everyday language, the various ideas which its terms evoke must be carefully distinguished from one another, and a new and more precise language must be developed in which each idea is represented by a specific symbol (e.g. word, sign, or name). Chwistek's method of sound reason expresses this nominalism. Its application reduces reasoning to the performance of purely mechanical operations upon symbols, analogous to calculatory operations. With the help of this method Chwistek hopes to secure the greatest amount of certainty in knowledge, within the limits of human reason. He insists there is no break in continuity between the kind of knowledge obtained in daily life and the kind obtained in the theoretical and experimental sciences. Consequently scientific propositions themselves are subject to the qualified certainty which the method of sound reason can give ; knowledge of the world can never be complete.[1]

Logic and Sound Reason

Because sound reason critically employed suffers from obvious limitations, Chwistek finds it necessary to supplement the uncontrolled operations of sound reason by a new device which he calls " logic ". However he uses the term " logic " in two distinct senses which he himself does not carefully differentiate. In one sense " logic " is taken to be " the basis of all thought " and can be construed as a general methodology. Its function is to distinguish the various categories of experience and to supply fixed rules in accordance with which sound reason may operate. With its help it is possible to differentiate beliefs which are held because of unreflective habit from those supported by reflective thought. It is with logic as a methodological instrument, co-extensive with the method of sound reason, that Chwistek has been concerned up to this point.

In the second and more frequent sense in which Chwistek uses the term, logic is identified with a formal system. A formal system must of course conform to the principles of sound reason already mentioned. However, the specific task of

[1] This fact is easily recognized if the conditions of human knowledge formulated in the criteria of sound reason are understood. Moreover, on the basis of the theory of types a large set of sentences must be regarded as meaningless and consequently cannot be admitted as knowledge.

formal logic embraces the formulation of concepts in terms of an unambiguous and precise symbolism as well as the elimination of appeals to intuition and of "metaphysical" assumptions. Consequently the development of a formal logic involves not only the analysis of the concepts of systems of knowledge already adopted, but their reconstruction on the basis of this analysis. In this way Chwistek hopes to avoid the hypostatizations against which his nominalism is directed, to expose the inadequacies of idealistic, realistic, and anti-rational systems of logic, and to convince the reader of their uselessness. Chwistek himself employs "logic" in this second sense only when he formalizes the mathematical sciences. However, although he does not apply formal logic to the philosophy of science and problem of reality, he offers constructive suggestions concerning these domains based upon his system of logic.

The construction of a system of formal logic [1] is carried through by specifying carefully directives of meaning, primitive concepts, axioms and rules governing operations. In consequence it is possible to determine almost mechanically whether an expression can be regarded as meaningful and whether a proposition can be regarded as logically valid. Although the procedures involved are highly formalized, and although no attention is paid to the referents of the signs employed, the results obtained conform in a rough way to those secured by less rigorous methods. In this way logic serves to supplement and control the unanalysed operations of habitual thinking.

Chwistek devotes the major portion of the present book to the construction of a logical system which will fulfil this task. His system is not yet complete since certain portions of mathematics have not yet been incorporated within it. Neither is his system entirely adequate since parts of it are not free from ambiguity.[2] Nevertheless Chwistek's belief that a completely satisfactory apparatus can be constructed remains unshaken and even in its present state seems to him to supply methods necessary for combating anti-rationalistic philosophies.

[1] Such a system will be called a formal system.

[2] These claims will be justified in the appendix which contains an exposition and criticism of Chwistek's system of formal logic (called "semantics" or "rational metamathematics"). This appendix contains material designed to aid the reader interested in the more technical aspects of Chwistek's work. The introduction, which is addressed to the more general reader, includes only a general account of the aims and methods of the system of semantics (cf. Section II).

II

Chwistek's Views on Logic and the Philosophy of Mathematics

It has been pointed out that for the most part Chwistek regards logic as a formal system [1] and it is his primary concern to develop such a formal system with the help of which it will be possible to derive various known portions of logic [2] and mathematics. For an adequate appreciation of the motives which led Chwistek to construct a *new* logical system (which he calls "semantics"),[3] it is essential to bear in mind recent developments in logic and the philosophy of mathematics.

Recent Developments in Logic and the Philosophy of Mathematics

Up to the nineteenth century mathematicians conceived their discipline as being exclusively the science of quantity. Kant, for example, regarded geometry as the study of quantitative relations of space. He claimed that the proof of geometrical propositions required a certain kind of sensuous, non-empirical, non-logical intuition (of space). Moreover he maintained that these proofs exhibit a constructive character, i.e. that they are based upon rules which stipulate the way in which the intuitions corresponding to mathematical theory must be constructed.

However with the development of projective geometry, which makes no use of metrical concepts,[4] it was soon realized that geometry might be conceived as dealing with non-quantitative relations. Additional discoveries, such as the principle of duality, geometries in which the validity of the theorems is independent of the kind of elements treated,[5] and perhaps above all non-euclidean geometries, led to the complete breakdown of the Kantian conception of geometry.

Geometry was now conceived as the study of certain abstract

[1] In this connection it should be recalled that a formal system is a system in which the directives of meaning, primitive concepts, axioms, or construction rules, and rules governing operations are precisely formulated. The theorems of such a system are derived by the application of the stipulated rules.

[2] As developed by other logicians.

[3] Or alternatively "rational semantics", "metamathematics", "rational metamathematics", "formal metamathematics".

[4] e.g. the notion of distance.

[5] e.g. the line geometry of Plücker and the sphere geometry of Huntington, in which the line and sphere respectively rather than the point were taken as the fundamental elements.

relations between unspecified (not necessarily spatial) elements. This new conception led to the application of postulational methods to geometry by such writers as Hilbert, Pasch, Veblen, Pieri, etc. Assumptions were explicitly formulated in order to make possible truly rigorous demonstrations of geometrical theorems, without any appeal to our intuition of space. Consequently the problem of the consistency of sets of geometrical axioms [1] received widespread consideration. It developed that the solution of this problem depended in turn upon the problem of the consistency of the axioms of arithmetic. In other domains of mathematics also, the attention of inquirers became directed toward providing a rigorous axiomatic foundation, with the consequence that a general study of postulational methods was inaugurated, a study which persists to this day.

During the nineteenth century foundations were also laid for the ultimate breakdown of the Kantian conception of arithmetic, as the science of quantity which depends upon sensuous intuition. The first important step in this direction was taken by Weierstrass and Kronecker, who maintained that the system of natural numbers is the basis of all branches of mathematics and that it is logically possible to arithmetize all portions of mathematics. They asserted that all mathematical entities can be defined in terms of the integers and that all mathematical results [2] can be expressed as properties of natural numbers.[3] The actual task of arithmetizing mathematics was undertaken by Cantor, Dedekind, and Weierstrass

[1] The axioms of geometry were formulated as propositional functions which contain the primitive or undefined concepts as the only variables. The only restriction imposed upon these variables is that they satisfy the axioms

At this point it may be well to recall several familiar logical distinctions A symbol with a precisely determined meaning is called a constant. The symbol "2", for example, is a constant. There are, however, symbols which have no independent meaning. Such symbols are called variables and the group of symbols, in which they occur, are called functions. For example, "x is a book" is a propositional function containing the variable "x". If, however, this propositional function is prefixed by the phrase "for all x", or the phrase "there is an x", it becomes a proposition. In the function "x is a book", "x" is called a "real variable". In the proposition "For all x, x is a book", "x" is called an "apparent variable". A set of axioms is called consistent if it is impossible to derive any two mutually contradictory theorems from these axioms.

[2] i e. mathematical operations and theorems.

[3] Kronecker even went so far as to advocate the elimination from mathematics as illegitimate, of all numbers other than the integers. A brief exposition and critique of Kronecker's method, which is based upon the concepts of congruence and modulus, is contained in Max Black: *The Nature of Mathematics*, New York, 1934, pp. 174–7.

among others, and led to increased rigour in the definition of the fundamental concepts of mathematics. Such notions as "limit", "area", "irrational", etc., were re-examined. New and precise definitions of these concepts were then given in terms of the integers and their relations, without any appeal to spatial or temporal intuition.

It was natural for mathematicians to suppose that if the fundamental concepts of the various portions of mathematics could be defined in terms of the integers it would be possible to unify all mathematics on the basis of elementary arithmetic. They assumed further that if they could axiomatize the various branches of mathematics, including elementary arithmetic, they could develop mathematics as a set of analytic propositions without any dependence upon intuition. The mathematicians therefore sought to axiomatize all parts of mathematics. Each part was constructed as a deductive system with its own set of primitive terms and a set of axioms concerning these primitives. The axioms were regarded as implicit definitions of these otherwise unidentified terms. This work was climaxed by the investigations of Peano, who, on the assumption that all branches of analysis had been rigidly formalized, axiomatically constructed, and reduced to elementary arithmetic, sought to complete the task of unifying mathematics by constructing an axiom system for elementary arithmetic, which would specify unambiguously the properties of the natural numbers. He constructed a system consisting of five axioms and succeeded in showing that from them it is possible to derive all the usual theorems of elementary arithmetic. His system contained three undefined concepts: "0," "number" (i.e. "integer"), and "successor". Unfortunately, however, it is possible to given an infinite number of interpretations of Peano's system which satisfy his axioms. As a matter of fact *any* serial progression whatsoever [1] satisfies these axioms.[2] It follows that Peano's axioms did not characterize the integers uniquely and that he did not supply the final basis upon which all of mathematics could be construed as a set of analytic propositions. It was

[1] The sequence of natural numbers is but one example of a progression which satisfies Peano's axioms. The sequence of even numbers is another illustration of a set of numbers satisfying these axioms. In this case "0" has its usual meaning, and the "successor" of a number is the result of adding 2 to this number.

[2] Cf. Bertrand Russell, *Introduction to Mathematical Philosophy*, New York, 1919, pp. 8–9.

thus necessary to supplement Peano's work by supplying an adequate definition of natural numbers. This task was accomplished by Frege and Russell.

Gottlob Frege surveyed the various definitions of "number", which had been proposed by his contemporaries. After a searching critique he concluded that "number" denotes neither subjective, spatial, nor physical properties, but that although it is a non-sensible attribute, it is nevertheless an objective one.[1] He fully agreed with the general tendency of mathematicial development to construct mathematics on purely rational grounds without any appeal to psychology or intuition. His standpoint was grounded on an analysis of the different contexts in which numerical expressions occur, from which he concluded that it is possible to define the numbers in terms of certain ideas [2] so general that they belong to logic.[3] On this view it is logic as the ultimate foundation which supplies the method for unifying all of mathematics.

Independently of Frege Bertrand Russell attained essentially the same results, and it is Russell's formulation in *Principia Mathematica* which has become most widely known.[4] Using four primitive ideas [5] and ten primitive propositions, only five of which are symbolical, he developed first the principles of logic and then the various portions of analysis.[6] In other words Russell (together with Whitehead) attempted to show *in full detail* that it is possible to reduce [7] mathematics to logic.

Unfortunately, however, certain difficulties may be raised in connection with Russell's system. In the first place the question of the consistency of his system is by no means settled by his assertion that it seems impossible to doubt or

[1] Cf. Bertrand Russell, *Our Knowledge of the External World*, New York, 1929, p. 218.

[2] Examples of such ideas are "implication" (i.e. "if . . . then"), "negation" (i.e. "not"), etc.

[3] i.e. he defined "number" as a property of a property of a collection. For present purposes it is sufficient to consider a collection as a set of objects.

[4] Russell's system cannot be called a formal system in the sense in which we have been using this term, since he did not stipulate all the rules of procedure to be used in deriving theorems.

[5] i.e. "elementary proposition", "negation", "assertion", and "disjunction".

[6] Amongst other things Russell was able to prove Peano's five postulates.

[7] In this context the process of reduction requires the definition of the concepts of mathematics in terms of logical concepts, the statement of the axioms and theorems of mathematics in terms of logical concepts, and the proof of these axioms and theorems by purely logical devices from the axioms of logic.

deny any of the principles of logic. As Hilbert [1] already pointed out, the problem of the consistency of a set of axioms requires consideration in its own right and Russell himself never gave any serious thought to this problem. Furthermore it is difficult to reconcile Russell's assertion on the one hand that logic is concerned with the real world,[2] and his insistence on the other hand that the laws of logic are true in all possible worlds [3] (i.e. that logical principles are relevant to a realm of entities which are not necessarily existent). Moreover other difficulties arising in connection with the platonic realism expressed by the latter point of view prevented Russell from offering any consistent view concerning the nature of classes.[4]

But for our purposes the difficulties which arise in connection with the systematic development of the theory of classes are most important. This theory, which serves as the foundation of modern mathematics, had been extended by Georg Cantor to include infinite (transfinite) classes as well as the usual finite ones. Unfortunately, however, it was soon shown that it is possible to develop a number of paradoxes within this theory. Typical of such paradoxes is the contradiction of Burali-Forti (151), the first paradox to be demonstrated within the theory. In order to understand some of the suggestions made to eliminate such obvious violations of the principle of contradiction, it is worth while formulating two of the simpler paradoxes, that of Russell concerning classes which are not members of themselves, and the paradox of the liar, sometimes called the Epimenipes paradox. Russell's paradox is as follows: If a class is conceived as a set of objects it is possible to form a class of such classes. It then seems to follow that certain classes include themselves as members. Thus, if non-men form

[1] Cf. p. xxx.

[2] Russell, *Introduction to Mathematical Philosophy, l.c.*, p. 169.

[3] *ib.* p. 192.

[4] Russell originally regarded classes as mere aggregates of terms or things. On this view the null class became a meaningless concept. In addition he soon realized that classes must have a different kind of reality than things, since in the course of his investigations he found it necessary to distinguish between a term and the class whose only member is that term. Consequently he abandoned his original conception of classes and advanced what he called the "no-class" theory. Since for practical purposes and under certain conditions functions of a function of a variable can be regarded as functions of the class determined by that variable, he maintained that it is possible to develop the theory of classes without ever using the concept of a class itself. On this view classes are but logical fictions, i.e. symbolic or linguistic short-hand devices.

a class, this class appears to be a member of itself since it is not a man (150–1). Bertrand Russell raised the question whether the class of all classes, which are not members of themselves, is a member of itself. Two contradictory answers can be given to this question. Where the symbol **A** is used to denote the class of all classes which are not members of themselves,

> if **A** is a member of itself, by definition it is not a member of itself;
> if, however, **A** is not a member of itself, it is a class which is not a member of itself, and consequently is a member of itself.

The paradox of the liar can be formulated as follows: When I say that I am lying,

> if I am lying I am telling the truth;
> if, however, I am telling the truth then I am lying.

Russell proposed to resolve these paradoxes by distinguishing three different kinds of statements: true, false, and meaningless.[1] He regards statements as meaningless when they fail to conform to a certain set of rules, which he calls the theory of logical types. These rules formulate the permissible ways of combining logical ideas. When Russell suggested such a set of rules he developed what has come to be known as the simple theory of types. In this theory a distinction is drawn between individuals,[2] functions which take individuals as arguments [3] (i.e. functions of type **1**), functions which take functions of type **1** as arguments (i.e. functions of type **2**), etc. In other words the type of a function is determined by its argument.[4] A class can be a member only of classes, not of any class whatsoever. Hence it is meaningless to speak of a class being a member of itself. Thus the statement in Russell's paradox are neither true nor false but meaningless statements, and it is impossible for the paradox to arise in significant discourse. It turns out, however, that while it is possible to resolve paradoxes such as Russell's with the help of the simple

[1] In this context "statement" is not to be identified with "sentence" or "proposition" "Meaningless" does not indicate a third truth-value.

[2] i.e. any object which is neither a function nor a proposition. Individuals are of type **0**.

[3] For example, in the function "x is a man", "x" is the argument to the function, which takes individual arguments only; in the function "R is transitive", the argument "R" will have only functions as values.

[4] Each logical function belongs to a single logical type. Moreover its arguments must be of the immediately preceding type.

theory of types, such antinomies as the Epimenides paradox cannot be eliminated by it.[1]

For this reason Russell proposed the branched or ramified theory of logical types. In this theory the type of a function is determined not only by the type of the arguments which it takes, but also by the form of the function. The theory is stated in terms of the notion of the " order " of a propositional function or of a proposition. A predicative function of an individual or a first-order matrix[2] is defined as an elementary[3] function of an individual. First-order functions are defined as functions whose arguments are individuals or are obtained from such functions by quantification.[4] A second-order matrix is a function which involves at least one first-order matrix among its arguments but has no arguments other than first-order matrices and individuals. Second-order functions are defined as second-order matrices or functions obtained from the latter by quantifying some of the variables, and so on for functions of higher order. An analogous hierarchy of propositions can easily be specified. It turns out that the branched theory of types is sufficient to remove all of the paradoxes which have been developed in the theory of classes.

Chwistek's Early Contributions to Logic and the Philosophy of Mathematics

It is in connection with the theory of logical types that Chwistek made his earliest contributions to logic and the philosophy of mathematics. His achievements were two-fold. In the first place he was the earliest logician to advocate anew the simple theory of types for the elimination of the

[1] As Ramsey pointed out Russell's paradox involves only logical concepts, while the paradox of the liar is based upon a non-logical concept, the concept of " truth " as well as logical concepts. Cf. Ramsey, *The Foundations of Mathematics*, New York, 1931, pp. 20–1.

[2] While in this introduction Russell's own statement of his views is followed, it is worth noting that "matrix" is a syntactical term (cf p. xxxvi, n 1) while "function" and "individual" are not. Strictly speaking therefore, predicative functions should not be identified with first-order matrices.

[3] An elementary function is a function which contains no quantifiers, i e. a function which contains neither the universal operator " for all (every) x " nor the existential operator " for some x ", or " there exists at least one x ".

[4] Quantification is the process of asserting a propositional function of all or some values of one or more variables. If all the variables are quantified the function becomes a proposition. If only some of the variables are quantified the function remains a function.

paradoxes of the theory of classes.[1] And secondly he seriously criticized the use of existence axioms in logic and mathematics.

As it happens the ramified theory of types does not permit the development of Cantor's theory without an additional assumption, known as the Axiom of Reducibility. This axiom asserts that every propositional function[2] of any order whatsoever is formally equivalent to some propositional function of order one.[3, 4] In his earliest consideration of Russell's work[5]

[1] It is perhaps worth while to point out that several other suggestions have been made for avoiding these antinomies Zermelo and Fraenkel, for example, maintain that the paradoxes need not occur if the axioms of the theory of classes are carefully formulated.

Other writers seek to eliminate the paradoxes by means of a distinction between an object language and its various "metalanguages" An object language is a language which is the object of investigation "Metalanguages" are of two kinds: syntactical and "semantical". The syntactical language is the language in which the forms of the sentences of the object language are studied. The "semantical" language is the language in which the relations between the symbol and the thing symbolized are investigated. For example, "It is snowing" is a sentence in the English language, which can be regarded here as an object language. The sentence "'It is snowing' contains three words" is a syntactical statement since it is concerned with the structure of one of the sentences of the object language and states a syntactical property of this sentence. The sentence "'It is snowing' is true if and only if it is snowing" is a "semantical" statement stating a "semantical" property of this same sentence of the object language. Hilbert's distinction between a language and its "metalanguage" corresponds to the distinction drawn here between an object language and its syntax language Professor Alfred Tarski initiated formal investigations in "semantics". Cf "O pojęciu prawdy w odniesieniu do sformalizowanych nauk dedukcyjnych" ("On the Concept of Truth in reference to Formalized Deductive Sciences"), *Ruch filozoficzny*, vol. 12, 1930–1, pp 210–11, and "Der Wahrheitsbegriff in den formalisierten Sprachen", *Studia philosophica*, vol. 1, 1936, pp. 261–405, a translation of a work which appeared originally in Polish. Both the syntax language and the "semantical" language can be constructed as formal systems and a theory of types can be specified for each of these languages Many writers, on the basis of Ramsey's distinction between the logical and non-logical ("semantical") paradoxes (xxxv, n. 1) make use of the (simple) syntactical theory of types for the elimination of the logical paradoxes and the "semantical" theory of types for the resolution of the non-logical paradoxes.

The above usages of the terms "metalanguage" and "semantics" must be distinguished from Chwistek's use of these terms, which will be considered below (xxxviii ff and Appendix) It is for this reason that these terms have been inserted in quotation marks in the present discussion.

Chwistek has never specifically commented upon any of these methods of avoiding the paradoxes. However, his general attitude toward existence axioms and toward the distinction between a language and its "metalanguage" (xli) are sufficient to indicate that he would emphatically reject any of these proposals.

[2] Of individuals as arguments.

[3] This assumption, which largely removes the distinctions drawn between the orders within the types, is necessary in the development of the theory of real numbers Two functions are said to be formally equivalent if they are (materially) equivalent for all values of the variables contained in these functions. Cf. p xlvii, n. 1.

[4] And similarly for higher types.

[5] *Z.S.*

an analysis of *Principia Mathematica*, Chwistek seriously questioned this axiom.[1] He suggested that the situation be remedied by retaining the branched theory of types but rejecting the axiom in question. This proposal involves the identification of classes with propositional functions and consequently the development of a modified theory of classes.[2] A number of years passed before Chwistek worked out in full detail the radical modifications required in developing a theory of classes subject to this far-reaching restriction.[3] He then pointed out the real difficulty in connection with the Axiom of Reducibility : it is not a proposition of logic but an existence axiom, with whose help it is possible to " prove that there are objects which perhaps cannot be determined " even though " to have any object it is necessary and sufficient to have a proposition from which this object is to be obtained by a wholly determined formal process ".[4] In the interim between Chwistek's proposal of the theory of constructive types and its actual construction, he suggested a return to the simple theory of types.[5] Unfortunately, however, this theory also depends upon existence axioms, e.g. the Axiom of Infinity, which asserts the existence of infinitely many individuals.[6]

The Meaning of Semantics

Thus while Chwistek clearly indicated his dissatisfaction with Russell's attempt to complete the refutation of the Kantian thesis concerning mathematics, he was also aware of the shortcomings of his own early views on the theory of types.

[1] As a matter of fact the grounds which Chwistek gave for his criticism of this axiom turned out to be utterly false, as he himself later realized. He asserted that the Axiom of Reducibility led to a contradiction within Russell's system What Chwistek actually demonstrated was that when a certain postulate of Poincaré is added to the axioms of Russell, a contradiction ensues. This postulate was formulated by Poincaré as follows . " Consider only objects which can be defined in a finite number of words." Chwistek has always accepted this postulate together with its implications.

[2] In this theory identity holds only between classes.

[3] Cf. *T.C T.* Chwistek called the new system which he developed the theory of constructive types, or the pure theory of types. It was characterized by the fact that in conformity with Poincaré's postulate it contains only a finite number of primitive symbols, and a long series of verbal directions for the construction of additional symbols and for the transformation of expressions in a *finite* number of operations. Its proofs are completely symbolic.

[4] *T.C.T.*, p. 10.

[5] *A.L.F.*

[6] Chwistek pointed out that alternative existence axioms can be assumed and that each choice of axioms leads to a distinct theory of classes.

Consequently he soon proposed a new formal system which he called semantics,[1] which he hoped would accomplish what Russell had failed to achieve. He wished to include in this new system those features of previous systems which he considered valid, and at the same time to avoid the difficulties encountered in these systems.

Chwistek felt, for example, that Bertrand Russell was on the right track when he attempted to reduce mathematics to logic. Yet in so far as the theory of logical types presented by Russell as a necessary concomitant of his system was not entirely satisfactory, Chwistek did not feel that Russell had achieved the end he had in mind, i.e. the unification of mathematics. Nevertheless he wished to retain not only Russell's general aim but the deductive method of presentation which Russell employed in *Principia Mathematica.*

Actually he went beyond Russell when he began to develop the latter's suggestions. In the first place even though Russell failed to derive mathematics from axioms which undoubtedly belong to logic, Chwistek still maintained that it is possible to unify mathematics, if not with the help of logic alone, then with the help of semantics.[2] In other words the system of semantics is the result of an extension of the logistic thesis; Chwistek asserts that mathematics *and* logic can be reduced to semantics.[3] In the second place he not only developed the

[1] The determination of the meaning of this term is the problem under investigation in the next few pages. Nevertheless it may now be said that whatever meaning Chwistek assigns to this term his usage must be distinguished from that of most contemporary writers on the subject. He does not intend to develop a theory of meaning, nor to enter into contextual analysis. In spite of his attack upon the hypostatization of entities (xxviii), he is obviously not seeking the referents of words. Moreover he is not concerned with the study of the responses of individuals to the names of entities He does not analyse the relations between the symbol and the thing symbolized, i.e. such concepts as "truth", "designation", "satisfaction", etc. This, of course, is not to say that he has no interest in any of these investigations. It is just that these projects are not the concern of semantics in his sense of this term. On his view semantics is concerned neither with psychological nor non-formal investigations. Chwistek explicitly rejects a distinction between a symbol and that which it symbolizes.

[2] i e. with the help of logical and semantical concepts.

[3] Evidence for this assertion can be found *only* in Chapter VII, where Chwistek's system is worked out rigorously. Chapters IV–VI should be considered as an attempt to familiarize the reader with Chwistek's technique and to supply the motivation for its introduction. Although a number of clues which turn out to be very helpful in understanding the later chapter are given in these earlier chapters, the proofs contained in the latter are intuitive. The gap between logical and semantical notions is not completely bridged and Chwistek's theory cannot be evaluated on the basis of this material.

system of semantics deductively but as a formal system. Thus the system of semantics is an attempt to define the concepts of logic and mathematics in terms of two primitive semantical concepts " * " and " c ", where all the construction and transformation rules governing semantical concepts are explicitly formulated.[1] The axioms and theorems of mathematics and logic are stated in terms of the primitive and defined concepts of semantics, and the axioms and theorems of these domains are proved with the help of the transformation rules of semantics. These proofs are purely symbolic.[2]

Henri Poincaré suggested [3] several rules for the conduct of logical investigations concerning the infinite. He advised [4]:

1. " Consider only objects which are capable of being defined in a finite number of words."

2. " Never lose sight of the fact that every proposition which concerns the infinite is a translation, an abbreviated statement of propositions which refer to the finite."

3. " Avoid classifications and definitions which are not predicative."

Well aware of the situations in which Poincaré's rules are of value, Chwistek adopts them in slightly modified form, formulating them in such a way that they can be applied to the concepts of semantics. It is characteristic of Chwistek's procedure that none of these rules appear either among the

[1] The sign " c " is called an expression (xl) as is any combination of the two primitive signs which is obtained with the help of these carefully stipulated construction rules.

[2] For further details concerning the technical aspects of Chwistek's system see the Appendix.

[3] Poincaré's suggestions were intended to achieve two objectives: first, the avoidance of the paradoxes of the theory of classes; second, the development of a general method for constructing mathematical entities, which will possess certain properties demonstrated to exist. The theorem which states that there is no greatest prime number is an example of this second point. It can be shown that this theorem is true, for if any prime number p is taken to be the greatest possible prime, and the product $1.2.3.5. \ldots .p$ formed from all previous primes, a new number p' can obviously be constructed by adding 1 to this product. This number, if it is not itself prime, is divisible by a prime which must be greater than p. It is possible to determine in a finite number of steps whether p' is prime. If it is not prime, it is possible to determine in a finite number of steps by which prime it is divisible. This proof does not actually require that this new number be calculated. As a matter of fact it would be impossible for any individual to carry out this proof for every possible case. Even for cases where p is relatively small and only a few steps are required to calculate the new number p', it would be very impractical to compute its value.

[4] Henri Poincaré, *Dernières Pensées*, Paris, 1913, pp. 138–9.

rules governing the system of semantics or in the system of semantics itself. He regards the first rule, which is an instrument for the criticism of classical Mengenlehre, as a special case of Occam's razor,[1] one of the criteria of sound reason. In as much as the entire system of semantics is constructed in conformity with the method of sound reason he finds no need to include this postulate among the rules of his system. As a rule governing the system of semantics, it obviously cannot be contained within this system.

Nevertheless Chwistek minimizes the negative role of this postulate, since he feels that it requires the modification and reconstruction of classical mathematics rather than the rejection of large portions of this subject. For this reason he gives a positive interpretation of Poincaré's first rule, an interpretation which is based upon the ambiguity of the word " word ".[2] He formulates this postulate as follows : " Consider only objects which are capable of being defined in terms of a finite number of expressions." Although this interpretation of Poincaré's principle does not prevent the construction of a formal system which would include mathematics,[3] it does impose definite restrictions upon any such construction. In the first place the concept " expression " must be a basic concept of any such system.[4] In the second place fundamental revisions are required in the classical theory of classes, particularly with regard to such infinite classes as the real numbers. Poincaré's second rule, which must also be interpreted as a restriction placed upon the construction of expressions,[5] is important in this connection, since it is applied in conjunction with the first postulate. Thus all expressions even those concerning infinite classes can be constructed with the help of a finite number of expressions.[6]

Chwistek's thesis that a theory of types [7] is essential for

[1] Cf. *T.L.*, pp. 125–6.

[2] This ambiguity exists in French and Polish as well as English.

[3] As Poincaré himself would maintain.

[4] Thus " expression " is a technical term of Chwistek's system of semantics and is therefore placed in quotation marks. Since in subsequent discussion this term will be used only in the sense indicated and since Chwistek himself does not employ quotation marks, they will be omitted in what follows.

[5] Since propositions are defined in terms of expressions.

[6] Expressions which can be constructed in a finite number of steps by the application of stipulated rules will be called constructible. This is not to say that these expressions must actually be constructed. It is essential only that it is possible *in theory* to construct them.

[7] Either the simple *or* pure theory of types.

the elimination of contradictions and meaningless statements may be regarded as a reformulation of Poincaré's third rule. Like the other postulates, and for the same reason, such a theory is not found among the explicit rules governing the expressions of Chwistek's system. Nevertheless, every statement of this system is constructed in conformity with such a theory.

In his investigations on postulational methods Hilbert found it necessary to study the structural properties of signs. Chwistek was apparently much impressed by this aspect of Hilbert's work. He points out that the great merit of the formalist school,[1] lies in its initiation of syntactical investigations concerning the properties of systems of symbolic logic. But he seems not to recognize that for Hilbert syntactical investigations constitute a field of study distinct from logical investigations themselves. Indeed, Chwistek asserted:

> " . . . Professor Hilbert assumes a system of axioms containing the principles of the Logical Calculus together with some purely Mathematical axioms (e.g. Zermelo's axiom); and he endeavours to prove with the help of " metamathematical " methods that they imply no contradiction. . . . Suppose he has proved by means of these primitive ideas and propositions " [of the logical calculus] " that a system of propositions (say p, q, r) is compatible with them. Then he has simply proved these propositions. If he has used (explicitly or tacitly) other ideas or propositions, then he has assumed some new hypotheses, which appear more general than Zermelo's axiom, etc. . . . Note that Hilbert does not assume the Theory of Types . . . such a 'metamathematic' cannot be essentially different from the Logical Calculus, this calculus being as a matter of fact a simple consequence of the laws of our thinking." [2]

In a footnote to this same paragraph Chwistek adds:

> " . . . there is a Meta-mathematic dealing only with the meaning of symbols, but never with the truth or falsehood of propositions. Therefore there is no means of providing a mathematical or logical proposition with such a Meta-mathematic." [2]

It would therefore seem that Hilbert's influence upon Chwistek's intellectual development has been more apparent

[1] Of which Hilbert was the founder.

[2] *T.C.T.*, p. 11. Spelling, typography, the usage of capitals and quotation marks conform to the original text.

than real except in regard to terminology.[1] In fact Chwistek's own procedure when he develops the system of semantics is the very antithesis of that suggested by Hilbert. He does not follow Hilbert in distinguishing between mathematics and "metamathematics"; thus, he does not construct one language whose concepts form the basis for the derivation of logic and mathematics, and a second language whose theorems state the syntactical properties of the first language. Rather he constructs a single language, which includes theorems concerning the structural properties of expressions in addition to logical and mathematical theorems.

We are now in a position to formulate more clearly what is meant by semantics. Semantics is a formal system which makes use of a carefully specified symbolism based upon two signs "*" and "c".[2] The sign "c" is called an expression. Any combination of these signs obtained by the application of carefully stipulated rules is also called an expression. It is evident that the expressions of semantics must be constructible in the sense in which we have defined this term. The symbols usually employed in logic and mathematics are correlated with certain of these expressions and are allowed to replace them.[3] Chwistek himself defines semantics as the "science of expressions". In other words semantics treats only those configurations of signs which are expressions in the sense indicated. Further rules are stipulated whose application to expressions yield new configurations called theorems. These theorems are of two kinds. Some of them are sentences of the language of logic and mathematics. The remaining theorems formulate syntactical properties of this object language. Nevertheless both kinds of theorems are derived by a single method.

It is clear then that Chwistek has attempted to achieve within a single system three distinct objectives, which may be briefly indicated by the rubrics "constructibility", "metamathematics", and "reducibility to a more general basic science". These elements are borrowed from the representatives

[1] i.e. Chwistek borrowed the term "metamathematics" from Hilbert. It should be noted that Chwistek's usage of this term in the passage cited differs from his usage in the current work. But each of Chwistek's usages differs from that of Hilbert.

[2] In Chwistek's eyes one of the lessons learned from the paradoxes is that they are caused by the ambiguities of the words used in everyday language. Consequently the various ideas evoked by words must be carefully distinguished and each idea represented by a specific sign.

[3] In practice these symbols are regarded as expressions.

of the three schools of thought engaged in the discussion of the philosophy of mathematics during the present century: the intuitionists or constructivists, the formalists, and the logisticians.[1]

III

The Theory of Plural Reality

Chwistek has been concerned with the problem of reality for many years and first stated his views on the subject almost twenty-five years ago in a lecture delivered before the Philosophical Society in Cracow.[2] A few years later the material presented at that time was published in considerably amplified and modified form.[3] Chwistek's views became the subject of heated debate in Poland, although the severe criticism to which they were subjected[4] was based on interpreting Chwistek's formulations without reference to the context in which they occurred.

Chwistek never intended his views concerning the problem of reality to constitute a new metaphysical theory. On the contrary he specifically stated that " the problem of the present study is the establishment of the meaning of the term 'reality' ".[5] His theory of plural reality must therefore be regarded as an attempt to specify the various ways in which the term " real " is used[6] and not as an attempt to provide a solution of the " philosophical " problem of reality. Accordingly it is a misinterpretation of Chwistek's intent to take literally his assertion that there is no one true reality but that

[1] This is not to say that Chwistek accepts all the ideas advanced by the writers in question. For example, he rejects Russell's theories of types because they depend upon existence axioms. He rejects Poincaré's opposition to constructions based upon the use of logical symbolism, and his stress upon the necessity of mathematical intuition (80).

[2] Cf. *T.L.*, pp. 139–151.

[3] Cf. *W.R.*

[4] Cf. e.g. *Przegląd Warszawski*, Rok 2, Tom 1, 1922, which contains a review of *W.R.* by Professor Tadeusz Kotarbiński, pp. 426–8, and an article by K. Irzykowski, pp. 291–306.

Przegląd filozoficzny, vol. 25, 1922, which contains a critique of Chwistek's views by Dr. Roman Ingarden, pp. 451–468.

Determinism nauk przyrodniczych (*The Determinism of the Natural Sciences*), by Joachim Metallman, Kraków, 1934, pp. 52–7.

[5] Cf. *W.R.*, p. 3.

[6] In conformity with this interpretation the phrase " concept of reality " has been used throughout Chapter X to render the word " rzeczywistość ".

there are at least four different realities. Moreover, it is not a relevant criticism to point out that on his analysis it is impossible to account for all aspects of experience. Chwistek himself is not interested in such questions as whether any of the well-known cosmological theories is the only theory which can account for the totality of experience. He notes, for example, that not only do both materialism and idealism seem valid to their adherents, but that to one and the same individual materialism may at one time appear to be the only valid doctrine, while at another time idealism alone may seem to explain certain portions of experience in a satisfactory manner.

This interpretation of Chwistek's theory of reality as an attempt to distinguish the different usages of the word "real" conforms to his general nominalistic approach (xxvi–xxvii). On his view the proper task of philosophy is definition rather than demonstration.[1] Accordingly, he regards the enumeration of the various ideas evoked by the term "real"[2] as genuine philosophical activity, whose object is the attainment of maximum certainty in knowledge within the limits of human reason. Sound reason, when applied to the problem of reality, therefore, prevents ambiguities and the confusions arising from them.

However in *The Limits of Science* Chwistek does not actually define the four meanings of the term "real" which he distinguishes; and he speaks of the "criteria" of the various realities without specifying them. What he does is to suggest four different contexts in which the word "real" is used. Thus he notes that "real" is a predicate employed in connection with four different kinds of entities: atoms, things and persons, images,[3] and sensations. Accordingly, he distinguishes four different "concepts of reality", that of physical reality, of natural reality, of the reality of images, and of the reality of sensations. In so far as each of the entities mentioned is characterized by a different set of properties, a different "concept of reality" is employed each time one of them is called "real". Although Chwistek himself does not regard this classification of the various "concepts of reality" as

[1] Cf. *W.R.*, p. 7.

[2] Chwistek asserts the importance of distinguishing the various meanings of a term because the use of the same term in different meanings, i.e. "operating with different patterns of reality" (270), is unavoidable in everyday language and because there are no criteria on whose basis it is possible to decide which meaning should be employed at a particular moment.

[3] e.g. dreams.

exhaustive it seems worth while to point out some of the more obvious omissions in his account. He does not, for example, make clear in what sense macroscopic objects are said to be "real". Neither does he state whether such entities as genes and chromosones are to be regarded as "real" in the same sense as are atoms. Nor does he specify in what sense certain numbers can be regarded as "real".

Nevertheless in his earlier writings on the theory of plural reality Chwistek did attempt to define the various meanings of the word "real". At that time he not only regarded *Principia Mathematica* as the model for all deductive systems, but he maintained that various portions of philosophy, in particular those portions which deal with reality, can be formulated as deductive systems. He called such formulations "formalizations", and maintained that their primitive concepts and axioms are not arbitrarily posited but are derived from an analysis of experience. These axioms contain concepts specific to the theory of reality as well as certain logical concepts. In stating these axioms,[1] ten in all, Chwistek employed six propositional functions as primitive.[2] He contended that certain sets of axioms chosen from this group implicitly define the different meanings of the word "real".[3] He suggests

[1] Cf. *W.R.*, pp. 30–4.

(1) If an object is given immediately it is real.

(2) If an object is visible, it is real.

(3) If an object is real, it is visible or given immediately.

(4) Certain real objects are not visible and are not given immediately.

(5) An object is visible if and only if it is visible during waking life.

(6) There are objects which are visible, which need not be visible during waking life.

(7) An object is visible if and only if it is visible under normal conditions.

(8) There are objects which are visible, but which need not be visible under normal conditions.

(9) Part of a real object is real.

(10) If part of an object is real, that object is real.

This set of axioms is a contradictory set.

[2] "x is real," "x is given immediately," "x is visible," "x is visible during waking life," "x is visible under normal conditions," and "x is part of y" (cf. *W.R.*, *l.c.*).

[3] Cf. *W.R.*, p. 33.

Sensational reality is defined by axioms 1, 2, 3, 5, 8.

The reality of images is defined by axioms 1, 2, 3, 6, 8.

Natural reality is defined by axioms 1, 2, 4, 5, 7, 9, 10.

Physical reality is defined by axioms 1, 2, 4, 5, 8, 9, 10.

The adequacy of Chwistek's definitions is not in question here although some of the problems raised by his analysis have already been indicated.

further that from each of these sub-sets important theorems concerning "reality" may be derived, although he himself did not derive any of them.[1]

The fact that in *The Limits of Science* Chwistek does not construct such formalizations might lead one to suspect that he has abandoned this method. However, in a recent letter (28th May, 1939) he states: "I have not abandoned this conception, although I think it has only theoretical importance." And indeed, in the book itself, Chwistek makes sufficiently clear that the formalization of reality is possible, though it requires to be based on semantical considerations. He points out that it is possible to construct symbolic representatives of the objects of experience, i.e. configurations of signs denoting these objects (268). He also contends that it makes no difference whether signs are interpreted as things, collections of atoms, visions,[2] or expressions (85). He suggests that it is possible to correlate with signs not only logical and mathematical concepts but philosophical concepts as well.

Chwistek's position on questions of logical theory have influenced the formulation of his views on the problem of reality. He requires, for example, the acceptance of a theory of types prior to the formalization of reality. With the help of this theory he distinguishes an infinite number of meanings of the word "real" in addition to the four meanings already indicated. For there are formalizations of higher type which take formalizations of lower type as arguments. However, this theory of types, which Chwistek calls "metascientific", is not formulated very precisely. Thus while Chwistek maintains that each of the four formalizations (i.e. each of the four "concepts of reality"), are of a different order although they

It should be noted that different concepts of reality are based on contradictory axioms. The concept of natural reality for example is based on axiom 7, which is the contradictory of axiom 8, upon which the concept of physical reality is based.

Chwistek maintains that the four "concepts of reality" thus defined are equally plausible and that it is possible to develop a consistent philosophical doctrine on the basis of each of these concepts. Consequently he asserts that no one philosophical theory is to be preferred to any other. Nevertheless Chwistek feels that a philosophical view based upon the concept of the reality of sensations conforms more adequately to the nominalistic approach than any other philosophical position.

[1] None of these deductive systems can be regarded as formal systems since their formation and transformation rules have not been explicitly formulated. Nevertheless they might be developed as formal systems.

[2] Visions are one kind of images.

are of the same type,[1] he nowhere sets up a precise hierarchy of orders. Nevertheless at very isolated points he does venture to make such comments as: the concept of physical reality is of higher order than the concept of natural reality (279). In spite of the lack of an explicit formulation of the "metascientific" theory of types this theory is of use in resolving some of the epistemological puzzles raised in connection with dreams. Chwistek maintains, for example, that it is not an error for an individual to regard his dreams as "real". Dreams are just as "real" as are persons or things. They are merely of a different order. On the other hand it would be wrong for an individual to regard the sensation which he experiences when he is dreaming as sensations of the same type as those which he experiences when he is awake. Chwistek's contribution to philosophical theory thus rests on the method he has devised by which it is possible to obtain precision of philosophical concepts.[2]

[1] Bertrand Russell's formulation of the branched theory of types is based upon a distinction between various orders of functions and propositions. First-order functions, for example, are defined as functions whose arguments are individuals. Although all such functions are not of the same type Russell pointed out that in practice these differences of type are neglected. (Cf. Whitehead and Russell, *Principia Mathematica*, vol. i, Cambridge, 1935, pp. 161–2.) Chwistek has therefore interpreted Russell as maintaining that although functions are of different orders, they are all of the same type (cf. *Z.S.*, p. 320). Consequently when Chwistek introduces a "metascientific" theory of types in connection with the theory of plural reality, he maintains that functions of different orders may be of the same type.

[2] In his considerations of the problem of reality Chwistek has on occasion alluded to general semantics (as distinct from rational semantics) and seems to suggest its importance in dealing with the problem of reality. He does not, however, specify exactly what he understands by the term "general semantics" and always returns to rational semantics, the system of semantics developed at length in this book, for hints to be applied in resolving this problem.

Typical of this procedure is Chwistek's insistence that it is possible to treat only "patterns of reality". For he never explicitly indicates what is to be understood by this term. Obviously the word "pattern" is not used in precisely the same way as it was employed in Chwistek's logical considerations (296, n. 2), since he maintains that it is impossible to obtain singular propositions such as "*a* is real" [where "*a*" denotes a particular object (268–9)], with the help of rational metamathematics or semantics. Nevertheless it is clear that a "pattern of reality" is some kind of a function since it contains a variable. The pattern "*x* is real", for example, contains the variable "*x*".

IV

The Philosophy of Science

Chwistek's interest in the problems of the philosophy of science, as manifested in his published writings, is comparatively recent. As a matter of fact his work in this field has been confined almost entirely to a critical account of some of the fundamental concepts of the natural sciences.[1] Semantical considerations and the theory of plural reality, as theories constructed in accordance with the method of sound reason have therefore had an important role in the development of, Chwistek's views on the philosophy of science.

Chwistek's Conception of Science

Chwistek views science as an attempt to develop a consistent rationalistic view of the world, based upon simple, clear " truths " derived from experience by the application of sound reason (3). He points out three distinct elements involved in scientific activity: classification, description, and explanation. Although none of them is sufficient to characterize the method of science completely, scientists do nevertheless classify, describe, and explain phenomena (3). Explanation is given in terms of laws usually numerical in character (25). For this reason scientists must specify a conceptual apparatus which will permit the simplest possible solution of concrete scientific problems. Accordingly economy is not an end in itself but is relative to the particular problems in which the scientist is interested.

[1] Chwistek does include positive constructive work on the special theory of relativity but his views on this subject will not be considered here. In the first place since his early consideration of relativity theory (*G.N.*, pp. 215–222), his position has undergone continual evolution (cf. *L.R.* and *P.P.R.S.*), and there is no way of knowing whether these views have attained their final formulation. In the second place the present formulation of Chwistek's views on this subject (242–252), which is obviously quite different from his early position, was originally written in English. Unfortunately certain passages are not entirely clear (cf. e.g. p. 245). Present world conditions have made it impossible to clear up these obscurities. Furthermore this introduction is concerned with material of philosophical rather than of purely scientific interest. It will therefore be sufficient to point out that in connection with the theory of relativity Chwistek is interested in deriving the Galilean transformation, which implies that it is impossible to detect by means of mechanical experiments uniform rectilinear motion with respect to absolute space, and the Lorentz transformation, which implies that two events, which are simultaneons for an observer at rest in a given frame of reference, no longer appear so to an observer moving relative to that frame of reference.

Chwistek points out that in spite of the fact that the scientist endeavours to describe and explain phenomena, he is not interested in finding their "causes". Such questions as: "Why does the earth revolve around the sun?", "What is an atom?", etc., are metaphysical rather than scientific; they lead to fruitless investigations, which do not extend the scope of our experience (11). The extension of our experience is one of the primary aims of the scientist and involves the possibility of prediction on the basis of scientific laws.

Chwistek also realizes that our scientific knowledge is not all-inclusive. Nature does make sudden jumps (51). It is frequently possible to give explanations of phenomena in terms of scientific laws only *after* their occurrence. It is therefore impossible in many situations to make predictions of a kind which would prove useful. For this reason scientific knowledge in particular, as well as knowledge in general, can never be complete (xxvii).

Finally in conformity with his insistence that science must be based upon the method of sound reason, Chwistek opposes the introduction of "metaphysical", "ideal", anthropomorphic, and "fictional" concepts into science (xxiv, n. 1). He also objects to the use of rough analogies, because the scientist is likely to forget that they are of value only as auxiliary devices and cannot be regarded as accurate representations of scientific facts.

Such is Chwistek's general conception of science, a conception which he works out with the help of a critical analysis of material drawn from the different sciences. However when he turns to particular problems of the philosophy of science Chwistek selects for consideration material taken from the natural sciences rather than from the biological or social disciplines. As a matter of fact he discusses the methodological problems of physics almost exclusively.

Measurement and Arithmetic

Chwistek points out that in physics events are abstracted from the totality of experience with the help of sets of numbers. Each such set is called the spatial representation of an event. This process of abstraction, i.e. of "formalizing reality" [1]

[1] This meaning of the term "formalizing reality" obviously differs from that employed elsewhere (xlv), in so far as it involves a *mathematical* representation of reality (238).

is a device used to predict (238). The apparatus employed in this process evokes images of reality, but it must not be confused with actual events. It is for this reason that Chwistek sharply criticizes those who take these " images " literally (239), i.e. those who identify " concepts of reality " with " reality ". On Chwistek's view mathematics supplies an adequate apparatus for representing the properties of subject-matter without requiring any " metaphysical " assumptions. The derivative " $\frac{dx}{dy}$ ", for example, which is employed by the physicist is not a mathematical fiction as some writers maintain. For the velocity, which is represented by " $\frac{x_2 - x_1}{t_2 - t_1}$ ", would in that case also have to be regarded as a mathematical fiction,[1] a conclusion few physicists would be willing to accept. On the other hand Chwistek regards both the concept " derivative " and the concept " velocity " as expressions. Since the physicist makes discoveries which " transform the surface of the earth " (69) with the help of simple operations upon velocities, etc., Chwistek feels that it is vital for the physicist to know how to use these expressions, i.e. to perform these mathematical operations upon them.

Chwistek treats as expressions even the numbers used in the process of measurement. The theorems of arithmetic are derived with the help of logical and semantical devices alone and Chwistek insists that they need *not* be verified by reference to experience. The actual process of measurement he regards as a " crudely defined activity " (255), since it is impossible to set up a one to one correspondence between the results of measurement and real numbers. A physicist interested in measuring the length of a table would not be satisfied with a single measurement but would make several. He would then formulate the results of these measurements as a series[2] of numbers.[3] He would say that the length of the table is represented by a number greater than the smallest number

[1] It should be recalled that for Chwistek the word " fiction " is a term of opprobrium. He has already indicated that the concepts of the calculus are not fictions, since they can be developed in terms of the concepts of semantics. He suggests that they be introduced in a purely formal way in such a manner that they involve only constructible expressions (321–3). He has indicated that the concept of velocity at a point can be analysed in terms of the concepts of the calculus (200–3).

[2] In this context the word " series " is used in a non-mathematical sense.

[3] In Chwistek's terminology as a series of expressions.

in the series and less than the largest number in the series. Chwistek sees no reason why the physicist should regard the average of these numbers as the "true" length of the table since more accurate results can be obtained by utilizing the inequality which stipulates the range of variation of the numbers obtained by measurement. For this reason he insists that the length of the table "corresponds"[1] in some way to *each* of the numbers obtained in the process of measurement and that slight differences between the results of measurement can be disregarded provided that two numbers are designated between which all the numbers obtained by measurement can be found.

Chwistek's Conception of Space and Time

Chwistek's concern with the methodological problems of physics leads him to examine the foundations of geometry. He defines geometry as an experimental science, which depends upon the measurement of segments (217), i.e. of distances. He does not, however, consider the physical procedures involved in making measurements,[2] but confines himself almost exclusively to a consideration of the logical difficulties encountered in the development of geometry as a purely mathematical science. Chwistek's considerations therefore bear upon matters which are of primary concern to the mathematician rather than the physicist, and he regards the decision between euclidean and non-euclidean geometry as not raising any problem for physics.[3] Since the constructions

[1] The meaning of the word "corresponds" in this context is not indicated.

[2] In Chwistek's elementary considerations his analysis of arithmetic assumes that the reader is familiar with the procedure of measuring segments (70). Chwistek himself has never analysed this procedure.

[3] Chwistek disagrees with Poincaré on this subject largely because the problems involved must be approached from the point of view of the physicist rather than from the point of view of the pure mathematician. Poincaré has recognized this fact but Chwistek treats the issues involved from the point of view of the mathematician. In any case the main difference between Chwistek and Poincaré with regard to the "conventional" aspects of this problem may be summarized as follows: Chwistek sees no problem at all in connection with the application of geometry in physics, because of the identity of euclidean and non-euclidean geometry when applied to limited areas; Poincaré insists that whether one applies euclidean or non-euclidean geometry in the solution of a definite scientific problem depends upon which system of geometry is most convenient.

Chwistek opposes Poincaré's "conventional" resolution of this issue mainly because he feels that "conventionalism" leads to opportunism and irrational philosophical doctrines (234), particularly with regard to social problems.

of euclidean geometry are identical with those of non-euclidean geometry within sufficiently small areas,[1] he sees no need to decide which geometry is to be applied to existential material. In a letter dated 20th July, 1939, he says, "If we speak about a part of space there is no problem at all as to what space is to be assumed, because all spaces are approximately euclidean . . . sound reason does not countenance ideal objects, conventions, and fictions." In view of this statement it would seem that Chwistek feels either that the construction of a non-euclidean geometry is an interesting mathematical exercise with no relevance for physics, or that the application of either euclidean or non-euclidean geometry in physics yields essentially the same results. Yet he also realizes some of the difficulties encountered in applying euclidean geometry to certain portions of physics. He points out, for example, that it is difficult, if not impossible, to give an actual illustration of parallel lines. To an observer stationed at some point of a railway track, the rails seem to meet somewhere in the distance. Although it is known that they never do meet it is impossible to directly experience this fact because the observer can never simultaneously be at the point of observation and at the "point of intersection". Thus euclidean geometry does seem to involve "fictional" objects after all (xxiv, n. 1). It is also possible to raise similar difficulties in connection with the application of non-euclidean geometry in physics,[2] since it is impossible to give an example of an experiential point or line.

Apart from his position concerning the applicability of geometry to existential material, the point which Chwistek emphasizes most strongly in his discussion of space is that

He fails to realize that all conventions are not arbitrary and that what Poincaré has in mind when he speaks of the importance of "convention" is practical convenience.

Moreover Chwistek's failure to provide a place for macroscopic objects in his theory of plural reality leads him to identify natural reality with physical reality, i.e. to treat persons and planets in exactly the same way as he treats atoms. Chwistek commits here an error analogous to that of certain nineteenth century philosophers, who, on the basis of the laws of mechanics, proceeded to argue the question of free will.

However it should be noted that while in *The Limits of Science* Chwistek opposes Poincaré's position, elsewhere he points out the importance of Poincaré's analysis. Cf. *U.B.*, p 4.

[1] This is a *mathematical* fact.

[2] Nevertheless Chwistek recognizes that many contemporary physicists do use a geometry in which there are no parallels. He recognizes the important role of "congruence" (234–5) in geometry, although he emphasizes the mathematical definition of this concept, rather than the physical process of measuring distances.

events are specified in physics by means of a co-ordinate system. But he realizes that this specification of the spatial co-ordinates of an event must be supplemented by the specification of a temporal co-ordinate as well. There is no " real " or absolute time, but only events which are verified by reference to experience. Experience teaches us that certain events are earlier and others later, and enables us to correlate (with the help of clocks) numbers with temporal events (238). The concept of time is thus an abstraction from experience which utilizes the apparatus of mathematics. In consequence, although the physicist employs the concept of continuous time in spite of the fact that we do not experience sensibly continuous time, actually " it is impossible to take sensual continuity seriously especially because the meaning of this concept is not known " (240–1).

In his discussion of time Chwistek also points out the necessity of reasoning in conformity with a theory of types. While an individual speaking about time must speak in time, Chwistek recognizes the importance of distinguishing these two uses of the word " time ". In other words the fact that the time *in* which an individual is speaking is of a different type from the time *about* which he is speaking must be taken into consideration in all discussion.

The Philosophy of Science and the Theory of Plural Reality

Chwistek's theory of plural reality is important in connection with many problems of the philosophy of science. It is therefore worth while to give at this point several illustrations which show the relevance of this theory to certain of these problems. The physicist's concern with the problem of motion leads him to study the motion of those bodies which Chwistek calls " things ". For this reason when the physicist is studying the motion of bodies along an inclined plane, his analysis depends upon the concept of natural reality. Similarly, since the physicist interested in the motion of atomic particles has incorporated some of the results of this kind of investigation into the kinetic theory of gases, Chwistek would maintain that this theory is based upon the concept of physical reality. Even the concept of the reality of images is of importance in physics because the use of a microscope depends upon some " image " in the mind of the observer, i.e. some mental picture of that which he hopes to see with the help of the microscope (286). Chwistek also points out that the concept of the reality of sensations has been used as the basis for the criticism and re-examination of the meaning of certain fundamental concepts of physics. In this way certain "idealistic" elements found in this theory have been eliminated. He has in mind here the re-examination to which such concepts as "position" and "momentum" (256–8, 283) have recently been subjected.

The Problem of Determinism

Chwistek has been concerned with the "problem of determinism" over a period of years. *The Limits of Science* contains the fullest statement of his views on this subject up to the present time. He is interested mainly in those aspects of the problem which concern physics. Accordingly, he recognizes the futility of attempting to apply his results to the "philosophical" problem of free will. Nevertheless his discussion is marred by such digressions as his attempt to introduce Fermat's last theorem as evidence for the deterministic point of view (254).

Chwistek begins his discussion of the problem of determinism by a criticism of the "classical" views on the problem. He points out that those "classical" physicists, who advanced a deterministic conception of the world based their views on the conception of an "ideal" reality. They therefore distinguished the "real" length of a segment from its actually measured lengths. In opposition to these scientists Chwistek maintains that the notion of successive approximations to this "real length" is unnecessary, since the length of a segment can be stipulated as a number to be found between two fixed limits. Chwistek also criticizes the identification of the concept of *determinism* with the concept of *predictability*. He maintains that in a given system, even if all the elements necessary for the prediction of an event were known, it would be impossible to predict all possible results which can be obtained by the application of the rules of this system. Nevertheless Chwistek holds that this system may be determined (260) and that it may even be possible to predict the particular event in question. He offers his system of semantics as a model of a determined system, in which, nevertheless, it is not possible to predict all the results which can be obtained by the application of its rules. Thus on Chwistek's view determinism must be distinguished from predictability. For on the one hand an event may be both predictable and determined; on the other hand it may merely be determined. It should be noted here that Chwistek employs the concept " determinism " as a primitive or undefined idea. He does not even define it implicitly with the help of a set of axioms.

Chwistek raises similar objections in connection with the views of the " classical " indeterminists. He accuses them of maintaining that it is impossible to obtain increased precision by successive approximations to the " real " length of a segment. Consequently it is not surprising that he rejects this " idealistic " conception.

Nevertheless Chwistek also rejects the view that *contemporary* physics is based upon indeterministic concepts. He explains the position of contemporary physicists on the problem of determinism as a reaction to the two " classical " (" idealistic ") views on this problem. He maintains, for example, that the criticisms of " classical " determinism do not establish the fact that a determinism without " idealistic " suppositions would in any way be objectionable (256–7). Consequently he sees no reason why physics cannot be based upon deterministic concepts. He maintains that such concepts as " position ", " length ",[1] etc., can be determined *on the basis of experience* within " sufficiently narrow limits " (258). Moreover entire *classes* of numbers, not particular numbers, satisfy the inequations which define these limits. In spite of the fact that Chwistek does not regard predictability as the defining characteristic of determinism, he maintains that only events which lie within these limits can be predicted. He asserts that if some day it should become possible to go beyond these limits the concepts " position ", " length ", etc., would acquire a new meaning, i.e. they would become new concepts for which different limits had been specified. This point is explained by means of the theory of plural reality. It becomes possible to go beyond the limits originally defined on the basis of experience, only if the process of formalizing reality [2] is taken a step further, i.e. if a new formalization of reality is set up. New concepts are thus defined with the help of this new formalization.

[1] It should be noted that these concepts have an experimental but not an absolute meaning.

[2] It should be recalled here that a " mathematical concept of reality " is involved here (xlix).

The Problem of Induction

Chwistek concludes his discussion of the methodological problems of science with a brief consideration of induction. With the help of several illustrations he makes a number of general observations concerning the nature of inductive reasoning, and enumerates some of the difficulties encountered in formulating an adequate theory of induction. Examples of unwarranted extrapolations make evident the fact that a factor relevant in one context where induction is applied, may be irrelevant in another. Furthermore no general rule can be given concerning the number of instances necessary for a valid inference to be drawn from them. In some cases inductive reasoning cannot be performed; in others a general conclusion can be inferred from a single event. Chwistek also points out that the validity of inductive reasoning depends upon the meaning of the concepts involved, and upon knowledge of the facts and of changes in the relation between them. Moreover the elements of guessing and emotion cannot be eliminated from inductive reasoning.

Chwistek, unlike most contemporary thinkers, is not interested in justifying the use of the inductive method. He is not, however, content merely to make such general comments as those enumerated above. Accordingly, he turns to the question whether or not it is possible to construct a general law or pattern for all inductive reasoning. The discovery of the answer to this question he regards as the real "problem of induction". He goes on to reduce this problem to the question whether or not reality can be completely formalized, in as much as his original problem and his new formulation of it both obviously involve the abstraction of certain elements from experience. Since, however, reality can never be completely formalized (261, 269), he concludes that it is impossible to give a general pattern which governs inductive reasoning. He also uses the increase in factual knowledge, which results from the extension of the scope of our experience by means of improved apparatus (266), as additional evidence in support of his position. It is thus clear why inductive reasoning can never be formulated in a pattern.

Chwistek maintains that the principle of *complete* induction, introduced as a rule of procedure in semantics, cannot be regarded as a rule of procedure in the natural sciences, since it is impossible to determine with any accuracy the "transition from any one case to the following". Since the application of complete induction is a characteristic feature of some of the natural sciences Chwistek proposes to justify the use of this principle by means of probability considerations (267); for he sees no problem in applying the calculus of probability to existential material (255).[1]

Chwistek's views concerning the methodological problems of science were developed in consequence of his aversion to the presence of "metaphysical" elements in science. He was therefore led to formulate some very unusual views on the philosophy of science, some of which require further development. His conclusions concerning the "problem of induction" are almost entirely negative. Unfortunately he does not consider any of the specific issues under contemporary discussion concerning the validity of inductive reasoning. It would seem, however, that the very fact that no general pattern can be given for this type of reasoning should make these issues even more acute for Chwistek than for other logicians, since he has attempted to apply a single method to all portions of philosophy and science. It should be noted finally that Chwistek has made no attempt to give an exhaustive treatment of the methodological problems either of science in general or of physics in particular.

[1] The consideration of these claims must be omitted, in part because of the technicalities involved and in part because of certain unclarities which result from the fact that this portion of the text was written in English. It has proved impossible to clear up these obscurities at the present time.

CHAPTER I

INTRODUCTION

1. We are living in a period of unparalleled growth of anti-rationalism. Exact thinking based upon the principle of consistency is the sacrificial goat to which all the disasters of our times have been imputed. The World War and all the orgies of domineering capitalism have been attributed to rationalism developed in accordance with the requirements of exact thinking. Exact thinking is blamed for drying up the sources of the sacred enthusiasm and for causing the emotional exhaustion of our epoch. Exact thinking, it is alleged, has become the source of the excessive growth of materialistic culture, as well as the shrinkage and sterilization of spiritual life. The demand for a new logic, for new laws of thought more suitable to the needs of spiritual life, has become the hobby-horse for a whole galaxy of obscure and false doctrines, from the revived dialectic of Hegel to pragmatism, universalism, and the phenomenology of Husserl.

These doctrines have arisen in many cases owing to widespread ignorance, while at other times they have been dictated by completely dishonest tendencies. Their source is the tragic disintegration of science over a period of years and the despair born out of a perception of the weakness of scientific procedure.

The history of the spiritual culture of mankind may be reduced to the struggle between faith in the creative power of exact thinking on the one hand, and doubt and powerless self-humiliation in the face of the irresponsible aberrations of fanatics who never attempt to solve any concrete problem and relinquish the pleasure of overcoming real difficulties on the other hand. This struggle has been carried on for centuries with varying fortune. But at present we have entered into a period of incredible abasement of science, a period of the noisy superiority of groups of puffed-up eulogists of irrational nonsense, who are leading mankind toward open crime and violence—as a rule unknowingly but often quite consciously.

2. Reflecting on this sad state of affairs, Professor Władysław Natanson writes :

> " Ought we not to regard it as evil for comprehensive science to give instruments of incalculable power to nations who have not

grown up to them morally? We have conquered the forces of nature but we have not conquered ourselves. As a consequence myopic egoism arises, and as usual disasters ensue; we are retarded, we are turned back. In silence with apparent equanimity science betrays its high mission. Science has much to say to the nations. When will it say it? When will it find inspiration and power enough to warn, to restrain, to convince?" [1]

I think that one cannot leave these disturbing questions without attempting to answer them. One must at any cost ferret out the source of the evil, reveal it in all its nakedness, and completely root it out.

It will be seen that the matter is much clearer than would appear on the surface for basically nothing but fear and general inertia prevent the solution of these problems.

3. Despite all efforts, inherited prejudices concerning the metaphysical foundations of science have not as yet been overcome.

The critical attitude, with which laymen credit great scholars is not sufficiently far-reaching. When Bruno Winawer, the author of many comedies, derided the philosophers, he contrasted them with the representatives of the exact sciences and called the latter creators of new forms of life on earth.[2] He did not, however, observe that these same scholars humble themselves before philosophers and desire at any cost to set themselves up as specialists who discover the bases of philosophy.

Winawer could maintain this view so long as he did not read the popular lectures of Schrödinger, the creator of the wave theory.[3] On reading them, he was startled by the mass of irresponsible phrases and crude analogies contained therein, which compared contemporary physics with the so-called new reality in art, and the electrons with separate human individuals. He became even more disturbed when I assured him that this is a common fact and similar cases can be cited by the dozen.

4. It is a fact that naturalists of the extremely critical type restrict themselves too often to detailed investigations in their own field and disregard the endeavour for a rationalistic view of the world. Naturalists have a peculiar foible: they

[1] Władysław Natanson: *Porządek natury* (*The Order of Nature*), Kraków, 1928, p. 159.

[2] Cf. his articles published in *Wiadomości Literackie* (*Literary News*).

[3] Erwin Schrödinger: *Science and the Human Temperament*, translated by James Murphy and W. H. Johnston, New York, 1935.

indulge in metaphysical prejudices and seek popularity in the name of doctrines which go far beyond the bounds of sound reason and exact thought. Unfortunately they have great influence.

Things have come to such a pass, that to talk to-day about the distinction between the representatives of pure science and the metaphysician is indeed difficult; for in the writings of famous mathematicians, physicists, astronomers, and biologists, abject surrender to the authority of deplorable and fruitless metaphysical endeavours is found. I shall give the following examples:

The famous German mathematician, Hermann Weyl, prefaced his book entitled *Raum, Zeit, Materie* (a work in which he endeavoured to include Einstein's theory in his system) with a philosophical introduction typical of a professional metaphysician of inferior quality. In his opinion it is a sad necessity that philosophy oscillates from system to system—a sad state of affairs which " we cannot dispense with unless we are to convert knowledge into a meaningless chaos ".[1] In other words, bearing in mind the tragic maxim, " All beginnings are obscure," [1] we are forced to build on uncertain foundations.

In the entire conception of the foundations of science offered by Weyl, one finds no trace of that modesty and unpretentiousness in the presentation of a theory which is worthy of a representative of the exact sciences. There is no recognition of that fundamental principle, that the point of departure in constructing a world view should not be a confused metaphysics, but simple and clear truths based upon experience and exact reasoning. Weyl entirely neglects the fact that physical theories are pure abstractions, which one cannot even regard as images of reality and that their rule reduces to this, that they make possible the systematic classification of phenomena as well as investigations directed toward the discovery of unknown phenomena. He ignores the fact that if philosophy is to be taken seriously it must restrict itself to a critical analysis of the relation of scientific theories to experience and cannot be the basis of these theories. He does not limit his ill-timed ambitions and seek the foundations of science; he prefers to immerse science in a chaos of paradoxes rather than to give up beautifully sounding, showy phrases.

[1] Hermann Weyl: *Space-Time-Matter*, translated by Henry L. Brose, New York, 1922, p. 10.

Metaphysical chaos marks the ideas of even those representatives of theoretical physics who consciously construct their theories from fictional elements having nothing in common with reality. They all seem to long more or less consciously for reality and they substitute their fictions in its place. Despite the explicit stipulations formulated in their introductions, they speak definitively concerning the indeterminism of the microphysical world as if this were some reality underlying the laws of physics. Thereby they operate very arbitrarily with the concept of meaning. Appealing to the fact that the smaller their error in measuring the position of the electron, the larger their error in measuring its velocity, and conversely, they affirm that under these conditions the concepts of the position and velocity of the electron has no clearly determined meaning. Often, however, they forget to add that from this point of view, the concept of the electron itself and in general the concept of the microphysical world has no determinate meaning. If then the laws of motion of individual electrons are not discussed, that is only because it is not desired to question seriously the electron fiction.

Similar misunderstandings are evident in the case of the astronomers. When they write about the expanding universe and simultaneously maintain the finitude of the universe, they approach these matters as they would the inflation of a rubber balloon. Eddington, the famous astronomer, accepts the presence within this theory of features so paradoxical that if he himself did not believe in the theory, such a belief would exasperate him.[1] There is involved in Eddington's view a very primitive realism, in which it is hard to detect a trace of those recognized restrictions upon which contemporary science is based. There is, of course, no real basis for dispute as to the legitimacy of auxiliary constructions; a genuine basis for disagreement is derived from the dubious pretension to knowledge of the essence of the universe, conceived in the image and likeness of the objects surrounding us.

Furthermore, disregarding the fact that their knowledge of the so-called universe is very fragmentary, astronomers would like to decide the question whether there is life beyond our globe and to give it a negative answer. In short, they would like to return to medieval geocentricism.

Reflection on these matters makes it difficult not to ask

[1] Sir Arthur Eddington: *Discussions sur l'évolution de l'univers*, Trad. et avant-propos par Paul Conderc, Paris, 1933, p. 31.

on what grounds philosophy could be disregarded and science considered the source of that revivifying word about which Professor Natanson writes.[1]

As to the biologists it must be acknowledged that they are weighed down by an oppressive mass of anthropomorphisms and irresponsible anthropological discourses which supply abundant food for philosophical dilettantism.

Driesch, the well-known biologist, is of the opinion that biology cannot do without the concept of entelechy whose internal correlate is the soul. This supposition leads him to accept some basic dynamic intellectual element which can be found both in the ultimate individual elements of a manifold as well as in the manifold itself.[2] Needless to say, here is involved an irrationalism which denies that which to-day still happens to be called scientific thought.

At the present time behaviourists carry to absurdity that important truth, affirmed by Mach,[3] that the psychic states of another individual cannot be the object of direct knowledge. The behaviourists proceed as if there were no difference between the phenomena of life and those of inorganic nature. They do not take account of the fact that so-called sympathetic attitudes or their lack are but reactions to phenomena around us, and are worthy of as much consideration as is given to sight or blindness. The fact that they take no note of these reactions leads to misunderstandings, which do not increase the authority of exact science.

Let us now contrast the metaphysician influenced by Husserl, Max Scheler, with the Dutch anatomist Louis Bolk. According to Bolk, man is a degenerate monkey incapable of normal development.[4] Man is therefore said to constitute an evident negation of life and an obstacle to nature. Evidently entirely in his element, Max Scheler takes issue with this theory. He invokes the spiritual life, culture, and art. He did not observe that Bolk's specialized knowledge makes him a dangerous opponent. Metaphorically speaking, on the level of such generalities, the contents of one empty bottle can be transferred into another *ad infinitum*.

[1] Cf. 1. 2. (In cross-references to the text the number before the decimal point indicates the chapter, the number after the decimal point the article)

[2] Cf. Hans Driesch . "Das Organische im Lichte der Philosophie," *Atti del V Congresso Internazionale di Filosofia*, Napoli, 1924, pp. 615–625.

[3] Cf. Ernst Mach : *The Analysis of Sensations*, translated by C. M. Williams, Chicago, 1914, pp. 33–4, 54.

[4] Cf. Max Scheler : *Philosophische Weltanschauung*, Bonn, 1929, pp. 35–6, 142.

5. Finally I wish to discuss the ill-considered reflections of Sigmund Freud, the eminent professor of neurology, on the theme " love thy neighbour ". While I am not an adversary of Freud and freely acknowledge him to be one of the greatest contemporary philosophers, his polemic against the commandment: " Thou shalt love thy neighbour as thyself," [1] is based upon primitive arguments which reflect a narrow, ultra-bourgeois view of the world. This is a classic example of the harmful consequences of lack of logical training. Freud accepts the method of sound reason. Unfortunately, however, it is accompanied by all the prejudices of the bourgeoisie and he neglects the fact that the limits of sound reason are much narrower than it would appear.

The principle, that it is more desirable for an individual to have than to give, is not as evident as it seems. It may be that extreme altruism, which depends upon making sacrifices for others, is a disguised form of egoism. However, this egoism differs so fundamentally from trivial egoism that to reduce both types to the same instinct must be regarded as an obvious excursion into metaphysics, worthy of Hegel or Bergson.

When this type of discourse is compared with a work of any philosopher in the tradition of positivism,[2] it must be admitted that it is the professional philosopher who manifests clear thought and a critical attitude. It must be admitted that in our day, the great tradition of the exact sciences has ceased to be dominant and that it has become difficult to establish the boundaries between these sciences and irrational error.

Undoubtedly terms differ in meaning in different contexts. It is also difficult to extract from the chaos surrounding us that which really deserves to be called pure science. But it does not follow that it is necessary to succumb to the lure of verbal phrases because they guarantee an apparently unified view of the world. It must be explicitly noted that metaphysics is not and cannot be a view of reality because it involves a fundamental error at its very root, namely the assumption that there exists knowledge other than that which is based upon experience and exact reasoning.

Historically the desire for such knowledge has always appeared when the great aims of rationalistic science have been

[1] Cf. Sigmund Freud : *Civilization and its Discontents*, authorized translation by Joan Riviere, New York, 1930, pp. 81–7.

[2] Cf. Eugène Dupréel : *Traité de Morale*, Travaux de la Faculté de philosophie et lettres de l'Université de Bruxelles, t. iv, Bruxelles, 1932.

wrecked on a reef of paradoxes. Actually these paradoxes indicate paths toward new discoveries and toward a new and more profound formulation of scientific questions. But the dullness and inertia which are characteristic of the human mind and which lead toward the easy path of anti-rational subterfuge have permitted neither enthusiasm nor creative effort.

Those representatives of science who have not lost faith in it have concerned themselves too little with the disturbing phenomena of life about them. Alarmed and uneasy they have confined themselves to a limited sphere of detailed investigations. As often as the scientists have prepared to dictate the laws of science to mankind, they have encountered internal contradiction. Confused and broken they have been forced to withdraw from the field of strife.

There have been many such gaps in the development of scientific thought. I mention only the discovery of the incommensurability of the side of a square with its diagonal, the paradoxes of Zeno of Elea, the Copernican system, the theory of gravitation, the critique of pure reason, non-euclidean geometry, the theory of Darwin, the paradoxes of the theory of aggregates, the experiment of Michelson, and Einstein's theory which is based upon it, the discovery of radium and of the quantum properties of radiation. On each of these occasions the scientists, dismayed by their results, retreated and left the fate of human culture to the discretion of individuals, who were unable to perceive the implications of these findings and therefore were inadequately equipped to carry on the struggle. It was at just such a time that Plato rose, and on similar occasions Hegel and later Nietzsche, Bergson, and their lesser followers achieved extraordinary success. They gave to disappointed mankind a narcotic of vision and phantasy which they substituted for exact thought. Society prepared to yield to the authority of chosen intellects and to abandon its critical attitude toward prevailing relationships. I do not deny that on occasion their ideas were worthy of admiration, yet I cannot refrain from pointed criticism of their procedure.

6. Rational criticism originated in Greece, but because it had too radical a form in its early stages, it led to extreme scepticism. The Greeks were unable to surrender their over-extensive epistemological ambitions, and therefore easily succumbed to a feeling of despair during periods of failure. The paradoxes of Zeno of Elea are known to have checked the

development of Greek mathematics. Similarly the discoveries of Heraclitus and Protagoras, which are correct in principle, are known to have completely checked belief in a science based upon experience.

To-day it is difficult to evaluate the social influence of the sophists, because their activities are known only from the writings of their admitted enemies. Nevertheless it is certain that they helped unmask false methods of reasoning, current in societies supported by tradition and the authority inseparably connected with it. It is well known that in such societies, the most diverse absurdities are held to be evident merely because they have been so regarded for generations. Sound reason is identified with the cultivation of these absurdities. Any opposition is held to be a bad error.

Undoubtedly the sophists went too far in their criticisms; nevertheless it must be pointed out that the Greek scientists could not refute their doctrines because they neither possessed the principles of exact thought nor knew its limits.

Socrates opposed the sophists in the belief that there exists a sphere of thought independent of our caprice; however, its domain was the world of universal, confused, and vague concepts, i.e. precisely those which even to-day are subject to individual interpretation. Consequently this belief was inevitably doomed to defeat.

Lev Shestov emphasized that great truth which was known to several earlier authors, that the arguments employed by the platonic Socrates are both cavilling and subjective. Yet in contrast to the arguments employed by the sophists they are held to be objective and absolutely true.[1]

Plato was undoubtedly a creative individual of the highest rank. But that which has always been regarded as his chief merit, namely the fact that he avoided arbitrariness and subjectivism through the use of universal concepts, was also his greatest error and the source of long-lasting stagnation.

Centuries elapsed before man rediscovered the proper province of exact reasoning. During this period, from the famous Plotinus to Hegel and the pragmatists, the platonic dialectic was the source of the mythology imposed upon objective science.

Yet it cannot be denied that it was Plato's disciple, Aristotle, who took the first step toward constructing a system of the

[1] Lev Shestov: *Le pouvoir des clefs*, translated by B. de Schloezer, Paris, 1928, p. 101 ff.

principles of exact thought. Aristotle's system was a fragment and was not free from fundamental errors. It was but the first step along a long and wearisome path. Unfortunately men have regarded Aristotle's work as perfect and for many centuries did not attempt to go beyond it. Even to-day it is still defended and passionately adhered to, although it is well known that its study is a useless requirement. For many centuries learned theologians imposed upon mankind as valid truths strictly proved by means of Aristotle's infallible system, doctrines which have nothing in common with exact reasoning. Actually they more or less consciously made the most of the obscurities and defects of this system.

Nevertheless the postulate of consistency, formulated by Aristotle as the well-known principle of contradiction (*Principium contradictionis*) [1] was a great triumph of rational thought.

It is possible to try to avoid this principle and to extricate oneself from the maze of contradictions which result from false assumptions motivated by utilitarian considerations. However, no one has had the courage to say that better reasoning exists, which need not be governed by the principle of consistency. Furthermore, this principle permits the discovery of errors in reasoning where lawlessness reigns.

The Russian philosophers of the Bukharin school condemn Aristotle's logic because it permitted the fiction of immutable concepts and supported the existing social order and the blemish of slavery.[2] They credit Hegel with having unmasked the prejudice concerning the immutability of concepts and with having opened the way to social progress.

This entire doctrine was caused by misunderstandings.

The postulate of consistency created a basis sufficient to overthrow a social system based upon slavery. The institution of slavery involves obvious contradictions, although the attempt is made to conceal them by more or less skilful phrases. On the one hand slaves are regarded as beings inferior in principle and essentially different from free men ; but on the other hand the right to sell free men into slavery is accepted. Aemilius Paulus abandoned Epirus to his mercenaries and sold one hundred and fifty thousand free men into slavery. Among

[1] Cf. Aristotle : *Metaphysica,* translated by W. D. Ross, Oxford, 1908, bk. iv (Γ) ch. 3–6, 1005*a*–1011*b*.

[2] Cf. Nikolai Bukharin · *Historical Materialism,* authorized translation from the third Russian edition, New York, 1925, pp. 23, 170.

those enslaved were keen, sensitive men who were highly developed mentally and ethically. The moment they were enslaved they became chattels subject to the unrestricted orders of their owners, who in many cases ruthlessly and cruelly took advantage of their rights. The money obtained was divided among the soldiers. Each soldier obtained no more than eleven drachmas. Plutarch wrote :

> " Men could only shudder at the issue of a war, where the wealth of a whole nation thus divided turned to so little advantage and profit to each particular man." [1]

These words of Plutarch are significant. It is seen that where cultivation of the emotions is not on a sufficiently high plane, the principle of consistency does not serve as a check.

Considerably before Hegel, the prejudice concerning the immutability of concepts was overthrown in practice, if not in theory, by the rationalists of the eighteenth century who prepared the way for the French Revolution. Undoubtedly the French Revolution developed from a rationalistic culture which was based upon exact thought associated with the spirit of mathematics and disdainful of all thought that was not clear and precise.

This culture was a phase of the development of the dogmatic rationalism of the sixteenth and seventeenth centuries which was based upon the belief that mathematical and natural methods make possible the discovery of the essence of all things. This belief resulted in a series of contradictory philosophical systems and the need for a critical attitude.

The strength of the pre-revolutionary philosophers did not lie in positive constructions, but in criticism based upon the methods of the exact sciences and extended to limits not reached in any previous period. The materialists overthrew the myth concerning the substantiality of the soul. Hume unmasked the false ambitions of dogmatic rationalism. Voltaire ridiculed the false pathos of the Middle Ages, the cult of the devil, and the unbridled licentiousness of the feudal lords. Montesquieu overthrew the belief in the apriori character of law and morality by a simple comparison of facts drawn from different times and different societies. Jean Jacques Rousseau overthrew the prejudice concerning the intellectual superiority of the

[1] Plutarch. *The Lives of the Noble Grecians and Romans*, translated by John Dryden, revised by Arthur H. Clough, Modern Library edition, a reprint of Clough's edition published in 1864, New York, p. 339.

privileged classes. The Encyclopædists robbed science of the mysterious charm of the book of the seven seals.

It is known that Saint Simon, a typical rationalist who accepted the critical world view of the pre-revolutionary epoch, was the creator of socialism. The positivist Auguste Comte who was greatly influenced by Saint Simon was the creator of sociology. Hegel's dialectic was not necessary to see the inconsistency of a social structure which was characterized by the oppression of the poor classes by the ruling class. The Aristotelian principle of consistency and cultivation of the emotions, based upon the doctrine of Christ, was quite sufficient for this purpose. That Karl Marx, the great defender of the wronged, was a disciple of Hegel and appealed to Hegel's dialectic was the result of a fortuitous concurrence of circumstances. Hegel's dialectic was a minor influence in the works of Marx and did more harm than good because it produced the illusion that Marx was concerned with the self-contradictory idea. Actually Marx was concerned only with the distress of the working-class and with the creation of a view of the world which would remove this distress. The dialectic of Marx does not differ essentially from the constructive methods of the exact sciences. Marx did not write on the "love thy neighbour" principle only because even at that time this principle was an outworn requisite, which could not restrain bestialized business men.

The positivistic doctrines of Auguste Comte strongly influenced nineteenth century scholars and produced a type of critical investigator who, like Newton, rejected such questions as: what is this? and why is it so and not otherwise? —because such questions lead to fruitless investigations. This type of investigator tries rather to extend the bounds of experience as much as possible and to formulate the laws governing phenomena. This doctrine, although radically altered during the course of the years, is maintained to this day. Thanks to the works of Mach, Poincaré, Einstein, and many other investigators has been produced the contemporary naturalistic view of the world, based upon the principle of the economy of thought formulated by Mach,[1] and the relativity of the principles of the theoretical sciences. It might be thought that these doctrines would lead to the victory of the ideas of tolerance and social justice. Actually this did not

[1] Cf. Ernst Mach: *The Science of Mechanics*, translated by Thomas J. McCormack, Chicago, 1902, pp. 6, 480–490.

happen. In this view of the world there remained certain weak points, against which occurred an anti-rational reaction leading to the present sad state of affairs. The source of this reaction should now be sought and definitively eliminated.

7. Hegel was undoubtedly the creator of contemporary anti-rationalism. His doctrines resulted from misunderstandings which were caused by a superficial knowledge of Kant's philosophy and a completely erroneous conception of mathematical analysis. Kant contrasted metaphysics, whose concepts necessarily lead to paradoxes, with mathematics, the sphere in which precisely defined concepts are employed. Kant's doctrine became a permanent acquisition of mankind. However, a century of work was necessary before it was understood that the primitive concepts of mathematics are as variable as other primitives and that their force lies only in the fact that their domain can be fixed by means of a precise symbolism. Hegel made this weak point in Kant's doctrine the basis of his dialectic. Hegel seized upon the fact that the mathematical analysis of the time depended upon the very confused concept of infinitely small increments or infinitesimals. He argued that there are and there are not infinitesimals ; therefore infinitesimals are undoubtedly contradictory objects.

On the other hand it is a fact, Hegel maintains, that mathematical analysis permits the discovery of the laws governing nature, while a consistent algebra is on the whole quite unproductive. It clearly follows that mathematical analysis owes its fruitfulness to the contradictions contained within it. But if such be the case, the fact that metaphysics leads to contradiction is not a defect but the guarantee of its creative power. Let us therefore cease to fear contradiction. Rather let us be courageous enough to accept its aid. We will then discover the essence of things which Kant had regarded as unknowable, namely the creative idea which contradicts itself at every step.

This view is obviously much more presumptuous than any view hitherto developed by mankind.

Paradoxes which from the beginnings of the history of intellectual culture had been regarded as fatal, suddenly became the source of great optimism and an instrument to be employed in obtaining knowledge of absolute truth. These doctrines offered so great a temptation that it can hardly be wondered that they gained many adherents and are found to-day in many different forms. I think that the fundamental duty of the representatives of true science is to combat this doctrine.

The style in which Hegel addressed his students clearly reflected popular mysticism. It might have been thought that a practical joker had cut parts of sentences out of scientific works and put them together arbitrarily in order to stupify and frighten mankind. Because of his great knowledge of human nature and his great talent for propaganda Hegel knew how to insert whole paragraphs amidst a mass of nonsense and pretentious poppycock.

He begins with the idea that to be and not to be is the same thing, whereupon the following explanation is added:

> "If we look more closely we find that a proposition has here been asserted which, carefully considered, has a movement by which through its proper nature, it disappears. But in so doing it does what must be held to constitute its true content, it undergoes Becoming." [1]

When the reader is convinced that he can never fathom the thoughts of the master and that he can only gather crumbs which are gratuitously thrown to him, he encounters the following doctrine:

> "God is known as Spirit, who duplicates Himself for Himself, but at the same time sublimates this difference in order that in it He may be in and for Himself. It is the task of the world to reconcile itself with Spirit." [2]

By this time the reader begins to realize that the master has disclosed to him the proper goal toward which he should strive at all cost. Unfortunately he does not know how to do this. He feels that he lacks courage and has no confidence in his own powers. The master then addresses him in an entirely new language, a language which is wholly colloquial, in fact just such a language as is employed by every cook and provincial shop-keeper. He says:

> ". . . this task is entrusted to the German world." [2]

A new epoch in philosophy began with Hegel and the German romanticists. It was characterized by the exploitation of the vagueness of concepts for purposes of the authors, the utilization of a conception of the universe in the interests of aristocratic reaction and the conscious use of this conception for

[1] G. W. F. Hegel: *Science of Logic*, translated by W. H. Johnston and L. Struthers, New York, vol. i, 1929, pp. 102–3.

[2] G. W. F. Hegel. *Vorlesungen über die Geschichte der Philosophie*, edited by G. J. P. J. Bolland, Leiden, 1908, Einleitung, p. 85.

cheap rhetorical effects. Schopenhauer vainly tried to unmask Hegel's method. But his voice was drowned by the plaudits of over-enthusiastic admirers. A phalanx of imitators soon arose. Hegelians appeared in almost all the countries of Europe and it is well known that all Polish philosophy during the first half of the nineteenth century developed under the influence of Hegel.

The plague of anti-rationalism cast into the world by Hegel spread to fantastic bounds. At first it was stifled by the tremendous growth of the mathematical and natural sciences; later as a result of the crisis in these forms during the nineteenth and twentieth centuries, it assumed new forms.

The first of these new forms was Friedrich Nietzsche's artistic conception of reality. Soon afterwards Bergson suggested the re-examination of the paradoxes of Zeno of Elea from which resulted emphasis upon a new faculty of knowledge called intuition. Bergson was influenced by the monstrous doctrine of the pragmatists. At the same time there began to spread in Germany the "scientific" philosophy of Husserl which attempts to create a new scholasticism in the name of the belief in absolute knowledge.

After the war, Europe was swamped by vast numbers of irresponsible anti-rationalistic systems which in the main yielded nothing new but were based upon cheap phrases which have been known for a long time. These systems helped create an atmosphere of depression and fear among the representatives of the exact sciences and had a more or less conscious influence upon their method of thinking.

On reading the arguments of Bergson, it should be observed that they are based upon a number of unresolved mathematical problems. While Bergson condemns the mathematical method and seeks to replace it by a better method he characterizes the latter with the help of mathematical concepts which are awkward and colourless.

Bergson relies upon *vague* and *variable* concepts as symbols of acts of a special kind. While these acts cannot be performed, Bergson perceives their possibility intuitively. Intuition is not to be identified with that "experience which arises from the immediate contact of the mind with its object, an experience which is disarticulated and therefore most probably disfigured".[1] Intuition is true experience, an experience which

[1] Henri Bergson: *Matter and Memory*, translated by Nancy M. Paul and W. Scott Palmer, New York, 1911, p. 239.

is higher than that which is called human experience. To have this experience it is necessary to give up certain habits of thought and even of perceiving and to place oneself at the *turn* of human experience. Bergson writes:

> " . . . there still remains to be reconstituted, with the infinitely small elements . . . of the real curve, the curve itself. . . . In this sense the task of the philosopher . . . closely resembles that of the mathematician who determines a function by starting from the differential. The final effort of philosophical research is a true work of integration." [1]

Because mathematics has not satisfied any of our desires, which in fact cannot even be formulated, it should be thrust aside. Instead of mathematics a new science which will be easier and more pleasant will be created. In lieu of struggling with the difficult problems of the theory of mathematical functions more attractive operations will be performed which meet no opposition, because no basis for opposition can be found. Anyone of a different opinion may be told that he is lacking in intuition and that he must feel his way intuitively. After a certain time he will either understand or pretend that he understands. In this way will be revived the tradition of the old magi, who held society by dangling before it secret and esoteric knowledge. Intellectual slavery from which mankind cannot escape is to be organized on a large scale.

These or similar thoughts must have been present in those minds which are excessively individualistic and which, influenced by Bergson, are inclined toward opportunism. They appeared openly and shamelessly in Anglo-Saxon countries as pragmatism (William James) and humanism (F. C. S. Schiller).

Both these doctrines skilfully exploited the positivistic conception of the economy of thought (Ernst Mach) which in conformity with the views of Bergson, they grotesquely caricatured.

The principle of the economy of thought [2] cannot be precisely formulated. In practice it reduces the aim of science to the construction of a theoretical system of concepts which would enable us to know the truth, i.e. to adjust ourselves in the real world as simply as possible. The theory of Copernicus may be cited as an example. However, actually the theory of Copernicus decided nothing because it can be maintained with equal

[1] Bergson: *l.c.*, pp. 241–2. [2] Ernst Mach: *l.c.*

right that the earth revolves around the sun and that the sun revolves around the earth. This latter view follows from a consideration of the relativity of all motion. Nevertheless the theory of Copernicus permitted us to describe in simple fashion the motion of the planets and later led to the Einsteinian principle of relativity.

Neither can the latter principle be regarded as an absolute truth, although it permits the construction of a much more unified cosmological system than the Newtonian system, and although it discloses phenomena never dreamt of before. As a whole the system of Einstein is much more complicated than that of Newton, but it is much more economical in dealing with the problems which present themselves. It is therefore seen that naïve simple economy is not sought for its own sake, but as a means for the attainment of knowledge of that which is before us.

The pragmatists did not take the trouble to think these matters through. All they said was that if economy is being discussed, value and therefore utility, and not truth is concerned. There is no concept of truth without the concept of utility. That which is useful is true. Neither pure thought nor pure knowledge exists. Always and everywhere the element of belief and individual want is decisive. Concern with matters which are not connected with life is fantastic.

The reality of every day is not true reality, writes F. C. S. Schiller.[1] True reality is created in accordance with one's needs. There may be a certain hesitation in accepting this theory because the supposition of the reality of the things and persons surrounding us is regarded as useful. However, if it is understood that the belief in the ability to create a new reality is much more useful, no hesitation with regard to accepting this belief unconditionally will occur.

According to Schiller, Protagoras was the real creator of humanism. Schiller does not interpret the doctrine of Protagoras as extreme scepticism but as a new type of metaphysics.

Schiller treats the difficulties which arise in connection with the fact of the existence of mathematics. He briefly remarks that the evidence and objectivity of mathematics is an illusion which results from familiarity with the postulates of mathematics, the frequently accepted belief in their practical importance and the fact that mathematics does not consist

[1] Cf. F C. S. Schiller: *Studies in Humanism,* London, 1907, pp. 220–1.

of a series of isolated truths but its truths form a unified and coherent system.

Schiller does not mention the fact that such truths, as twice two is four, cannot be denied without considering the system of which they are a part, although in many cases it might be to our advantage to do so. Certainly everyone would deny that there is an advantage in accepting a system of arithmetic where $2 + 2 = 5$ if he has to pay someone else, but where $2 + 2 = 3$ if he is to be paid.

Contradiction would indeed follow from this, but in many cases contradiction has proved to be very useful and often cannot be distinguished from reasoning which is based on feeling and emotion. Consequently if on the grounds of utility, mathematics is generally regarded as an objective science, perhaps by the former term is meant a utility which has nothing in common with individual criteria, a metaphysical utility which is inaccessible to ordinary intuition. But utility in this sense, in so far as it is not simply a synonym of what Schiller calls truth, is a mere phrase.

Schiller does not refrain from employing underhand demagogic tricks. He fails to consider arguments in which rational criticism is employed and deals only with the naïve idealism of certain English metaphysicians. Consequently his arguments seem ultra-intelligent and effective. It is not strange then that the superficial reader, especially one who is seeking a quick and easy solution of the problem of knowledge does not see this.

The influence of the doctrines of Schiller and James has been much greater than is apparent. I quote from them but rarely, although their ideas live on in the works of many post-War German metaphysicians and adversely influence the minds of exact investigators the world over.

It seems to me that it is a waste of time to argue with them. I think that it is sufficient to emphasize the nonsense involved in their doctrines. It is much more important to confront them with a consistent world view which has been developed by the use of a critical and rationalistic method and which involves no metaphysical suppositions. The first step toward the attainment of this goal must be a consideration of the foundations of logic and of the question whether it is really possible to construct a system of logic which involves no metaphysical suppositions.

8. The doctrines of the phenomenological school, which was founded by the late Edmund Husserl, are representative of a certain type of anti-rationalism.

Husserl did not oppose science but desired to supplement it by a scientific philosophy based upon the conviction that absolute knowledge is possible.

I quote here the criticism of the doctrines of this school which I made in the introduction to a previous essay.[1]

> "The fact that the phenomenologists themselves repeatedly show the absurdity of their belief in the possibility of absolute knowledge spurs them on and forces them to unparalleled efforts, which prevent the development of their most characteristic views and prevents their progress beyond the sphere of pleasing conventionalism. Nevertheless the sin of verbalism, sanctioned by the deplorable 'pure grammar' of Husserl shakes this school to its foundations. Not only do the phenomenologists fail to attain the heights of which they have dreamt but they are brought back to the muddy dells of reality. . . .
>
> "Pure grammar is the means used by the naturalists and the naturalistically orientated epistemologists to prevent a critique of the bounds of everyday language. These investigators gave careful consideration to the meaning of philosophical questions which are apparently innocent and natural. Thus they initiated the exact investigations later conducted by the logicians, and in particular by the famous Bertrand Russell. Among other things they were concerned with such eternal questions as : What is truth ? What is matter ? What is man ? What is the good ? What is a work of art ? and so forth.
>
> "Those scientists who had failed to consider these questions, lost contact with reality and entered the sphere of fiction. The chief problem of the phenomenologists was to be the rebuilding of this contact."

It was to be re-established by an investigation of the real content of concepts, which does not differ essentially from that of which Plato dreamt. Different methods were applied but they were just as arbitrary and confused. Husserl began with a cavilling criticism of the nominalism of Hume[2] and pointed out that Hume did not show how it happens that certain ideas are produced by certain words. This objection is obviously childish because much more complicated phenomena are involved here than the phenomenon of gravitation, for example, about whose essence we neither know, nor hope to know, anything. Accepting independent concepts to establish meanings is very much like

[1] *T. V. M.*, pp 48–9.

[2] Cf. Edmund Husserl: *Logisch Untersuchungen*, Bd. II, Teil 1, Halle, 1913, pp. 184 ff.

positing that the weight of matter is located at the centre of the world in order to establish the law of gravitation.

The positive work of Husserl explained nothing because of necessity it was based upon arbitrary and confused assumptions. His works are filled with such assumptions.

It is clear, for example, that for Husserl the word *something* has a simple meaning.

> " The experience of an idea which is consummated in understanding the word is undoubtedly a construction, but its meaning is without a trace of being compounded." [1]

This distinction between the meaning of the word and understanding it is the result of verbalism and arbitrariness, since it is impossible to discover anything other than the ideas which present themselves when the use of the word *something* is being considered. These ideas might be called the meaning of this word. The phenomenologists maintain that only by the use of their hypotheses can the relativism which makes science impossible be avoided. Actually it is the phenomenological method which makes science impossible because it makes science depend upon some special faculty which has nothing in common with either reasoning or experience and which can not be controlled. If the method advocated by the phenomenologists were employed, sooner or later esoteric knowledge of the type found in the Ancient East and the intellectual and material slavery associated with it would recur. The pre-War essay of Reinach,[2] a disciple of Husserl, plainly manifested this tendency.

The Bolshevik revolution and its unexpected success destroyed the social illusions of the phenomenologists and transformed them into the obvious anti-rationalism of Hitlerian insanity.

9. The negative aspect of both positivism and materialism is that on the basis of these doctrines it is impossible to fix even approximately the boundaries of the exact sciences. In particular it is difficult to define the special status of mathematics. The old Kantian argument which depended upon the thesis that mathematical truths are certain while those of nature are approximate was until recently universally accepted. Even Poincaré took it seriously. The voices of those naturalists who

[1] Husserl : *l.c.*, p. 296.

[2] Cf. Adolf Reinach : " Die apriorischen Grundlagen des burgerlichen Rechtes," *Jahrbuch fur Philosophie und phänomenolo ische Forschung*, Bd. i, teil ii. 1913.

observed that arithmetic is not to consider the changing world (Le Dantec), had no great influence. For the most part attention was focussed on the fact that the world of geometry is an ideal world and differs fundamentally from the sensual world. The attempts of John Stuart Mill to regard points, lines, and planes as hypothetical objects proved to be unfortunate. This was also true with regard to Mach's attempt to discover correspondents of these constructions in the sensual world. It is clear that geometry does not depend upon inquiries of this kind and those who have advocated apriorism based their views upon this fact.

The reduction of arithmetic and geometry to the principles of formal logic, which was attained by Whitehead and Russell at the beginning of the century, was the crucial moment in the attempt to fix the boundaries of the exact sciences.

If the attempt to construct a great system of logic from which all the apriori sciences could be derived were successful, completely new perspectives would be opened up to science and an adequate basis for a critical and rationalistic method would be attained. A system of logic which permits mathematical theorems to be proved without the aid of the intuition of the creative individual by mechanical operations, which can be performed by any one who can understand ordinary arithmetic, was sought. The attainment of this ideal would have been so great a triumph for science that in comparison with it the attempts of the irrationalists would seem like child's play. It was to be expected that the representatives of radical criticism would have accepted the work of Whitehead and Russell with enthusiasm. However, the exact opposite actually occurred.

Peano's earlier attempt to formulate the apparatus of concepts and axioms of mathematics had already evoked a violent reaction on the part of Poincaré. Actually he both feared to break with the positivistic tradition and mistrusted the reaction of an extremely critical mind toward a work which had many weak points. Peano's apparatus of concepts was still far from perfect and it was possible to ask whether it would not lose its force the moment it was desired to mechanize it completely.

The second crucial moment in the development of this line of thought was the discovery of the paradoxes which follow from Cantor's theory of aggregates. These paradoxes were involved in the foundations of the new logic. Russell succeeded in removing these paradoxes by means of his famous theory of logical types but he was able to do so only by introducing certain metaphysical suppositions which a critical mind could not accept.

In the first place it was necessary to presuppose the existence of individuals which could not be further characterized. The existence of these individuals was an integral part of the system but no example of them could be given. In other words the domain of logic became an abstract world similar to the platonic world. The primitive concepts of logic became platonic ideas because they had to be explicitly distinguished from the signs by which they were introduced. Finally it was necessary to accept an additional hypothesis which assured the existence of infinitely many individuals. Otherwise finitism could not be avoided. On the other hand, if this hypothesis were accepted the existence of objects not definable in terms of the concepts of the system would have to be accepted.

In short it must be admitted that the system of Whitehead and Russell is such that either it does not contain the class of natural numbers or it contains a class of real numbers which contains as a sub-class numbers not definable in terms of the concepts of the system. The latter consequence which at the same time leads to the affirmation of the existence of the actual infinite evoked a particularly vehement reaction on the part of Poincaré. Poincaré was a decided nominalist and could not become reconciled to the existence of indefinable objects, much less to the existence of infinite classes of such objects. Poincaré regarded his belief as the fundamental postulate of a nominalistic logic. He formulated this postulate as follows : " Consider only objects which can be defined in a finite number of words."[1]

Poincaré thought that by proposing this postulate he invalidated the entire construction of Whitehead and Russell. Most of the adherents of the new logic were of the same opinion. This fact clearly shows the extent to which science depends upon philosophic views.

Mathematicians were divided into two groups. The members of the one group called themselves empiricists and the influence of Poincaré upon them is clearly observable. I think that they should be called nominalists. The nominalists rejected systematic logic, were satisfied with mathematical, intuition, and confined themselves to a verbal characterization of the intuitive method (Brouwer). This method differs from that employed in constructing a precisely defined system. The idealists, who from the point of view of the medieval tradition should perhaps be

[1] Henri Poincaré : *Dernières Pensées*, Paris, 1913, p. 138.

called *realists*, constituted the second group. The members of this group, relying upon Cantor's theory, failed to mention the system of Whitehead and Russell and restricted themselves to intuitive attempts to demonstrate the consistency of the axioms of mathematics (Hilbert).

With the mathematicians so divided in their opinion the expected rebirth of the exact sciences on the basis of a great system of logic which fixes their boundaries failed for the moment.

But the game was not finished.

Further investigations showed that the metaphysical suppositions of the system of Whitehead and Russell can be eliminated by basing the construction of a consistent system of logic upon a pure theory of types and upon the science of expressions, formulated symbolically, which I have called semantics. In other words the additional suppositions made by Whitehead and Russell are unnecessary.

Thus a new system of logic which satisfies the nominalistic postulate of Poincaré and which is compatible with the spirit of critical rationalism was developed. In spite of the extensive restrictions of this system it is no poorer than the system of mathematics which is based upon the axioms of Zermelo. Consequently it is adequate to develop all the material which is desired by most mathematicians.

When this new system is completely worked out, we will be able to say, that we have at our disposal an infallible apparatus which sets off exact thought from other forms of thought.

The old dream of the logicians concerning a consistent logical apparatus will no longer be a mirage. Just as now we have calculating machines, in time we will have the apparatus which is necessary to derive the general theorems of semantics.

However, I think that there is no reason to wait until this ideal has been achieved.

The very confirmation of such a possibility offers weapons which are adequate to combat the attacks of the anti-rationalists and to free us from any possibility of attack by them.

A science which is based upon an infallible system of logic and which involves no irrational assumptions will be able to fulfil the mission toward society which Professor Natanson requires of it.[1]

Such a science will not fall into error and will not be brought to a standstill as a result of its own illusions.

[1] Cf. 1. 2.

Such a science will be able to say to the nations :

Construct new concepts if nothing else, but guard against operating arbitrarily with them. Remember that otherwise chaos and error will result, and that it is possible to avoid them only with the help of a complete system based upon the principle of consistency.

Have the courage to search the obscure hidden corners of your system and do not be ashamed to admit that you were following the wrong path if from your assumptions you derive conclusions which contradict these assumptions. Do not believe that exact analysis necessarily leads to inertia and the depreciation of the imagination and emotional life.

The fact that recently a nation with a great cultural tradition has been mastered by brutal, ignorant individuals shows only that this nation was permeated by an irrational metaphysics.

History teaches that ultimately victory has always been the destiny of societies who employ the principles of exact reasoning. Exact analysis depraves only weak and inept individuals who find it too difficult for them. It should not be feared by young and healthy societies. They will always find sufficient strength to act upon thoughts which were obtained over a period of years by means of exact analysis and to work out a well rounded fruitful life on the basis of these thoughts.

CHAPTER II

THE LIMITS OF SOUND REASON

1. The conclusions obtained by means of sound reason must be distinguished from the *popular view* of the world. The popular view of the world or what is called " common sense " embraces " the things which *seem* obvious and inevitable . . . in a given society ".[1] The popular view of the world is a definite metaphysical system whose principles cannot be precisely formulated but which work quite successfully through the operation of habits. It is well known that the popular view of the world is always associated with escapism and is a synonym of banality and mediocrity. Crime and cruelty which seem to be necessary ingredients of the world are its inseparable companions. The fight against this metaphysics is difficult and dangerous. Its power of suggestion is so great that the only alternative to it seems to be a break with rational thought and reliance upon emotion and will, to decide everything.

Nevertheless history shows that even a serious struggle with dominant types of social relations, carried on in the name of irrational catch-words, does not involve a complete abandonment of sound reason. An individual may be ruled by a great passion and dominated by the expectation of a miracle to solve his problems; but it does not follow from this that when crossing the street he is to disregard traffic or that if there is no fuel, a motor will not run. These are indeed common-places and it may seem a waste of time to be concerned with them; however that may be, the facts to which they point form the basis of all man's intellectual creativity.

Sound reason functions smoothly in accordance with fixed rules, provided that men remain in familiar spheres and conduct themselves in accordance with ideas sufficiently well established to enable them to resolve their problems. Under these circumstances animals as well as men behave in accordance with certain mental habits which are generated independently of their will. But if an individual finds himself in a novel situation to which his conceptual apparatus is not obviously applicable, serious difficulties arise. Logic has therefore two

[1] A. E. Heath, " Reflection and Common Sense," *The Philosopher*, vol. xii, 1934, 16.

fundamental problems. The province in which the criteria of sound reason function in accordance with rules must be clearly distinguished, and systematic methods for avoiding the confusion of these criteria with thought-patterns dictated by habit must be indicated. These problems are much more difficult than they seem at first sight. There are no absolute criteria; there is no basis for maintaining that what is to-day regarded as a manifest truth may not some day become doubtful.

Human thought does not develop continuously and it is not possible to remain content with the piecemeal addition of new truths to old ones. Such a piecemeal accretion can continue only until an intellectual revolution takes place, with a consequent inversion of values. Thus by *sound reason* is to be understood the method for attaining truths which are not subject to intellectual revolution. While it is certain that there are such truths, they are primitive and banal. It is hard to suppose that they may become the basis for the construction of a conceptual apparatus, by means of which the laws of the development of science can be stated in precise formuli. Nevertheless it will be seen later that this is possible. But to solve this problem a long series of very tedious investigations must be conducted and many difficult problems connected with the individual sciences must be solved. Not until then will the technique necessary to overcome the bad habits of common sense be obtained. This task must begin with a critique directed against every-day language which, in effect, embodies the common sense dominant during a given epoch. Criticism of scientific language will be undertaken later and the foundations of mathematics, the most primitive and at the same time most mystical of sciences, will be examined. The history of the struggle of sound reason will have to be investigated together with the metaphysical myths encroaching upon its sphere. The task of rebuilding the foundations of mathematics and removing its metaphysical suppositions must be undertaken to convince ourselves that this impressive intellectual structure is the product of a procedure which is as simple and obvious as that of systematically laying brick on brick and which is involved in man's progress from the construction of wooden sheds to the construction of skyscrapers and stratospheric balloons. It will be shown that all else is but ornamentation and would even be amusing were it not regarded as the essential creative power and did it not conceal the fundamental mechanism of the science. It will be ascertained that in other domains as well, a similar fight against the misconstruction of this ornamentation and its restraining influence must be carried on before systematic construction work can begin.

The attempts of true science may be defined by means of two terms: *criticism* and *construction.* It has been customary to employ the terms *analysis* and *synthesis*, but because they have for centuries been connected with metaphysical myths their use involves a whole series of misunderstandings. The term *synthesis* is particularly dangerous because it has frequently been misused by the anti-rationalists. According to them synthesis does not denote a well-built structure but is a hastily put together scaffolding whose shaky walls are effectively hidden by paint. Neither does the term *analysis* as used by the philosophers have a precise meaning. It does not signify incisive criticism directed toward reaching the heart of the problem, since such criticism must be accompanied by construction. On the other hand, since it is necessary to rely upon something, if it is not recognized that there can be no permanent basis for opinions, other than the sort of reflection required for constructing a shed or a swallow's nest, our judgments will inevitably be based upon some prejudice. All this critical work will then have been in vain.

The dangers connected with superficial criticism will first be considered.

2. The critique of customs instituted by the sophists was radical and thorough, but because it was not accompanied by a constructive effort it was fruitless and pernicious. It will be seen later on how the results of this critique have been exploited. The reaction in antiquity to the activity of the sophists was relatively moderate and served to establish firmly the existing state of things. I am referring to the platonic dialectic. Plato was strongly influenced by Socrates but from his writings no clear opinion can be formed about the latter. In any case it is certain that Socrates was the first philosopher to make use of the method of sound reason; this constitutes his greatness. Socrates recognized that all thought is of the type employed in daily life and asserted emphatically that truth may be attained by the common as well as the educated man. This insight had the mark of genius although in its practical application it was warped by the confusion of sound reason with common sense.—In addition to his emphasis upon sound reason, Socrates entertained the conception of the perfect good. This conception is irrational and cannot be reconciled with sound reason. Men

recognize almost intuitively the difference between good and bad because they have been taught to make such distinctions since childhood ; nevertheless such evaluations are superficial. It is not clear what is to be understood by the perfect good and attempts to define this concept have failed.

It was the error of Socrates that he attempted to support the irrational idea of the good by the criteria of sound reason. Lev Shestov, in writing about Socrates made the following statements :

> "The problem of Socrates involved creating something from nothing and getting something from nothing." [1]

> "If one can speak in this way Socrates drinks his good as ordinary men drink water. He apprehends it with the eyes of the intellect, he grasps it with the hands of spirit. Just as things of the external world have a real existence for us, so the good has a real existence for Socrates." [2]

A more forceful account of the irrational elements involved in Socrates' thought can hardly be imagined. His attempt to establish his ideas by means of sound reason appears to be more dangerous and strange, the more convinced we become of their irrational character. It is readily recognizable that such an attempt will ultimately be disastrous to the operation of sound reason since the criteria of sound reason will be identified with those of common sense.

The way in which this confusion of concepts occurred can be shown by an analysis of Plato's dialogues.

Sound reason can operate in the sphere of crude and primitive definitions. These definitions are adequate only so long as we confine ourselves to familiar every-day phenomena. Failure to recognize this fact produces the illusion that every-day language is a perfect instrument which can be made to function smoothly in accordance with fixed rules once its use has been learned. The reverse is in fact the case. If we leave the sphere of the familiar events of daily life, difficulties are encountered with which the criteria of every-day language are unable to cope. Plato showed accurately how this happens. He begins with the notion that the concept of the good, of beauty or of love is simple and noble and it is necessary to reflect but a little in order to become convinced of this. Gradually this naïve view is replaced by a feeling of doubt and uncertainty which arises

[1] Lev Shestov : *Le pouvoir des clefs*, translated by B. de Schloezer, Paris, 1928, p. 8.
[2] Shestov : *l.c.*, p. 9.

because of the dialectical subterfuges to which the honest Socrates must resort in order to arrive at the desired result. It seems that the soul cannot be a harmony because the good of the soul would then be a harmony of harmonies[1]; but this is impossible. However, the suggestion is made that perhaps the harmony is better at one time than at another so that in the former case the soul itself is more virtuous. Whether or not this is the case, it is clear that these considerations involve uncertainty and confusion. The soul is discussed as if it were a fixed object although it is not clear what the nature of the soul is.

A child thinks of the human spirit as something like a breath which issues from the body. Although he may forget this naïve idea he will retain the general conception of something called the soul. Plato's Socrates speaks of the soul as if he were speaking of an object in every-day use, yet no one of his hearers considers that perhaps the term *soul* denotes absolutely no object. Those participating in the discussion readily agree that the soul cannot be compound, because its component parts cannot be specified. Apparently they do not realize how such an admission leads to a proof of the immortality of the soul. If something is not compound it could neither be created nor could it cease to exist. Indeed if it is difficult to imagine how something which is not compound can be created, it is even more difficult to imagine how something which is compound can have a beginning in time. In particular it is not easy to see how a hen comes from an egg, but Plato ignores such questions as trivial.

Plato's arguments greatly influenced the development of human thought. It is hard to discover why. Undoubtedly a proof of the immortality of the soul was very desirable from the point of view of religion; on the other hand, the thesis that the soul is uncreated is obviously inconsistent with religious beliefs.

The fact that all these matters were discussed in every-day language is very interesting. The product of the struggle for existence, a crude conception of creation, began to be regarded as absolute knowledge which goes beyond the bounds of the human imagination. As a result of the disconcerting paradoxes of the sophists and the ensuing despair of and disbelief in our ability to discover a firm foundation for beliefs, the fiction of pathetic and perfect knowledge was produced. It was supposed that this knowledge unravels the mystery of existence.

[1] Plato: "Phaedo" 77–8, 91–3, *The Dialogues of Plato*, translated by Benjamin Jowett, 3rd ed., New York, 1892, vol. ii.

At the same time there developed an aversion to thorough criticism as an instrument of Satan which leads to dangerous consequences.

To this day there are many naïve imitators of the platonic Socrates who ask simple questions in assured tones in order to force their world view upon those participating in the discussion. I am certain that Socrates was not so devious in his methods. Nevertheless I think that Shestov was right in regarding Socrates' motto: "I know only that I know nothing" as a mask hiding arrogant pride.[1]

Socrates' fundamental error had a fateful influence throughout the ages of history through Plato's pupil, Aristotle. The illogical character of the platonic dialectic was too striking for any one to be able to persuade himself that it conformed to the principles of sound reason. A systematic basis for perpetuating the confusions between the principles of sound reason and those of common sense was supplied by Aristotle. Nevertheless his achievements are noteworthy. His greatness lies in the fact that he formulated the fundamental principle of sound reason, namely the *principle of contradiction.* Aristotle recognized the importance of his discovery, calling this principle "the most indisputable of all principles".[2] It can be formulated in various ways. The present discussion will consider only the following formulation:

Two propositions, one of which is the contradictory of the other, cannot both be true at the same time.

It should be noted that in spite of its simplicity and triviality, this principle is valid only under certain conditions. It is valid only if it is invoked in the consideration of material to which our concepts are applicable.

> "To every definite question as to whether an object has this characteristic or that, we must respond with a yes or a no. As to that there can be no doubt whatever. But how are we to answer when an object is undergoing a change, when it is in the act of losing a given characteristic or is only in course of acquiring it? A definite answer should, of course, be the rule in these cases likewise. But the answer will not be a definite one unless it is couched in accordance with the formula 'Yes is no, and no is yes'; . . ."[3]

[1] Shestov . *l.c.*, p. 230.

[2] Aristotle · *Metaphysica,* translated by W D. Ross, Oxford, 1908, bk. iv (Γ) ch 4, 1006*a*, l. 4.

[3] G. Plekhanov: *Fundamental Problems of Marxism,* edited by D. Ryazanov, London, 1929, p. 114.

These words characterize precisely the situation. A determinate character is ascribed to a changing object. It is not said for example that such an object is both good and not good, both red and not red, both great and not great. Answers to definite questions must be completely definite. It is admitted that in such cases change is concerned. On the other hand the answer is too general to be satisfactory and it cannot be taken, as the partisans of the Marxian dialectic suppose, as the basis for precise scientific investigations. In the exact sciences the concept of a fixed object is supplemented by the concept of an object in phase so that the object may be good in one of its phases and bad in another. In this way the principle of contradiction is maintained. It is clear, however, that to achieve this, tedious and prolonged constructive work is necessary. Consequently the principle of contradiction when applied to any and all problems and investigated in every-day language must be regarded as false.

This state of affairs does not diminish the merit of Aristotle. The moment it is recognized that there is no exact science without the principle of contradiction and that all attempts to go beyond it are either only apparent denials of it (Marxian dialectic) or sterile anti-rationalism (Hegel, Bergson, etc.) the greatness of Aristotle's discovery must be acknowledged. Just because the principle of contradiction does not always function in accordance with the prescriptions of common sense, it serves as an especially powerful means for distinguishing common sense from sound reason. However, it should not be thought that reasoning in conformity with the principle of contradiction never violates the criteria of sound reason. The history of science has shown that the principle of contradiction may lead to a barren formalism, which without involving us in asserting falsehoods can give a distorted view of creative activity in science. This matter will be considered in detail later on.

The too general formulation of the principle of contradiction by Aristotle led to an erroneous conception of its function. In consequence the discovery of the actual role of this principle in sound reason required a long period of great intellectual effort.

These issues were obscured by Aristotle's scientific activity because of his naïve trust in common sense and his attachment to the existing state of affairs which is evident, for example, in his acceptance of slavery as natural.[1]

[1] Aristotle: *Politics*, translated by Benjamin Jowett, Oxford, 1908, book i, chs. 3–7, pp. 30–8.

R. D. Carmichael says:

> "Aristotle was almost entirely concerned with establishing what has been conceived already or of refuting error, but not with solving the problem of the discovery of truth... He thinks of confirming truth rather than of finding it." [1]

To attain this objective only a powerful dialectical-rhetorical apparatus for silencing the most skilled opponent was necessary. Aristotle's theory of the syllogism was such an instrument. This mistaken and thoroughly sterile doctrine was nevertheless a great discovery; it was the first attempt to mechanize reasoning and was the first example of a system of formal logic. However, it involved an unprecedented growth of verbal metaphysics which still flourishes and hampers the development of science.

Professor Tadeusz Sinko emphasized the dialectal-rhetorical character of Aristotle's logical works in his remarkable treatise on Greek literature.[2]

Sinko comments as follows on the *Categoriae*:

> "But who knows whether Trendelenburg (*De Aristotelis Categoriis*, Berlin, 1833) did not understand the matter more accurately when he tried to show that Aristotle's categories depended simply upon distinguishing the different parts of speech (nouns, verbs, conjunctions, and others)."

He has the following to say on the *Topica*:

> "The eight books of the *Topica* survey different well-worn helps useful in disputation on proposed themes and the evaluation of probable assertions."

Finally Sinko says:

> "The *Sophisticis Elenchis* which recapitulates the *Topica*, shows at length how the logic of the *Organon* mainly serves the needs of rhetoric."

Even in the famous *Analytica* a verbalism which leads to fundamental errors is clearly marked. Aristotle uniformly neglected the fact that uncritical acceptance of every-day language must lead to contradiction.

By admitting all the expressions in common use as equally legitimate, Aristotle was made powerless to cope with the paradoxes of the sophists and was condemned to evasions and subterfuges. It is scarcely possible to speak of having entered

[1] R. D. Carmichael. *The Logic of Discovery*, Chicago, 1930, p. 6.

[2] Tadeusz Sinko: *Literature grecka* (*Greek Literature*), tom. 1, część 2, Kraków, 1932

the sphere of precise knowledge if only such propositions as: *it is good, it is not not-good,*[1] etc., are considered, for their meaning even at first glance is quite confused. As has been pointed out, in such circumstances Aristotle could be certain of victory in debate[2] but such success does not amount to a solution of the problem of consistent thinking.

In short the theory of the syllogism in which but a few misleading phrases were employed, was of no value in investigating vital problems. Moreover it included a false principle of inference, i.e. the principle of the conversion of universal categorical propositions.[3]

According to this principle, from the premise

All devils are wrongdoers,

follows

Some wrongdoers are devils.

This premise is unquestionably true since by definition the devil is a wrongdoer. On the other hand this conclusion involves the affirmation of the existence of devils It can therefore be supposed that the existence of devils can be proved. But in similar fashion it would be possible to prove the existence of whatever was desired. Even the existence of square circles, glass mountains, and invisible wishing caps could, for example, be proved.

Some logicians attempt to defend this obvious error by maintaining that Aristotle did not allow for the possible existence of null concepts.[4] But this is precisely the difficulty of verbal philosophy. Null concepts cannot be neglected because it is never known in advance whether or not a given expression denotes such a concept. For every given case it must be shown that the concept is not null. A completed logic is necessary for this task if it is not explicitly assumed that a non-null concept is being employed. But the acceptance of such an assumption involves a fundamental reformation of syllogistic theory. Once the assumption of existence is explicitly made, verbalism is taken at its face value and the illusion that verbal logic is fruitful disappears once and for all.

Consequently the theory of the syllogism does not satisfy the

[1] Aristotle: *Analytica Priora*, translated by A. J. Jenkinson, Oxford, 1928, bk. i, ch. 46, 51*a*.
[2] Aristotle *l.c.*, bk. ii, ch. 19, 66*a*
[3] Aristotle: *l.c.*, bk. i, ch. 2, 25*a*.
[4] Jan Sleszyński: *Teorja dowodu (The Theory of Proof)*, tom i, Kraków, 1925, p. 40.

requirements for a purely formal system, namely that it should be an aid in solving interesting problems.

3. The history of syllogistic theory shows that even from a purely formal point of view the doctrine was never more than trivial. It therefore gave rise to the conviction that logic is unproductive.

Nevertheless Aristotle's spirit continued to flourish among the verbal metaphysicians. They are incapable of evaluating the conceptual apparatus of the exact sciences and satisfy their desire for a lasting foundation which will strengthen their authority by postulating the existence of eternal concepts which are concealed behind the expressions of every-day language. They regard every-day language as a perfect system which carries within itself the seeds of absolute truth.

However, just the reverse is in fact the case. Every-day language is not a consistently worked out system and never will be one. Professor Delacroix justly called it "a mixture of conventions and logic, of arbitrariness and reason".[1]

This naïve confidence in every-day language is one of the most curious and interesting of human ostentations. It reflects not only man's virtues but also his weaknesses with all their erratic manifestations.

Professor Rozwadowski, the great linguist, writes :

> " I said that language and languages are masterpieces of human culture ; undoubtedly, but in them there are also masterpieces of almost grotesque monstrosity. The situation here is on a par with such phenomena of human culture as the compulsory mutilation of the feet in China, compulsory tattooing of the skin for decoration, compulsory fantastic hairdress, and the deformation of the skull in strange ways by different human races." [2]

All these problems are completely ignored by the verbal metaphysicians. They stubbornly believe that if they use every-day language with sufficient accuracy—and it should be recognized that they identify accuracy with pedantry—they will be able to obtain entirely univocal definitions. In this way a bombastic and dull style is cultivated which produces the illusion of precision. Viewed objectively the outcome is trivial and futile.

An example of such an unfortunate method is the university

[1] Henri Delacroix : *Le langage et la pensée*, Deuxième Édition, revué et complétée, Paris, 1930, p. 10

[2] J. Rozwadowski : *O zjawiskach rozwoju języka* (*On the Phenomena of the Development of Language*), p. 9.

textbook of logic by Professor Kotarbiński. I cite the following paragraphs:

> "In the mouth of Peter, for example 'I' denotes Peter, while in the mouth of John, John is denoted. In the mouth of someone in Warsaw who utters the proposition 'Stanislaus lives here' the word 'here' frequently means that the one who lives 'here', lives in Warsaw, and conversely, etc." [1]

> "Similarly 'I understand a given name clearly' means that I am conscious of the properties which constitute its connotation. If therefore I always observe their presence in a given object, I will recognize it as the designate of this name, and conversely I determine that a given object is not the designate of this name only if I always observe that it lacks these properties." [2]

The author has obviously striven for accuracy, but in effect he has only substituted involved and clumsy expressions for simple and trivial ones, thus creating the illusion that he is employing univocal language.

Professor Delacroix says:

> "Words have many meanings. We always have more ideas than words and more words than ideas. Even in the language of a child, words describe complex situations and this complexity increases with the enlargement of experience." [3]

The point involved is that even in the case of the simple names mentioned by Kotarbiński, *horse, sparrow, five-grosz piece, cigarette,* for example, there is no precisely determined field of application so that no one understands them in such a way that he can always unambiguously decide what the designate of a given name is.

That such names as *five-grosz piece* and *cigarette* are vague is evident, whenever a coin with a partially effaced inscription or a cylindrical capsule filled with tobacco of an ungraded variety is examined. The names of species of animals also seem to be vague as soon as an attempt is made to apply them to a more or less developed foetus or to fossil remains. If it is admitted that every man must have a mother and that every such mother must be a human being, it has apparently been proved that men have always existed, although this conclusion contradicts the

[1] Tadeusz Kotarbiński: *Elementy teorji poznania, logiki formalnej i metodologji nauk* (*Elements of the Theory of Knowledge, Formal Logic and the Methodology of the Sciences*), Lwów, 1929, p. 24.

[2] Kotarbiński: *l.c.*, p. 27.

[3] Delacroix: *l.c.*, p. 216.

experimentally supported conclusions of the geologist. Consequently if it is desired that a determinate meaning be assigned to the word *man,* it must be assumed either that the first woman sprang from the rib of a man or that the mother of some man may have been a monkey.

Such questions involve us in such instructive paradoxes as those of Eubulides concerning the bald man and the heap of sand, or the paradox of Zeno of Elea about a rustle.

When one grain of sand is dropped it does not make a rustle. If no rustle is produced when a fixed number of grains is dropped, apparently no noise will be produced by increasing the number of grains by one. Consequently on the basis of the principle of complete induction a bushel of millet cannot make a rustle.[1]

It should be observed that even the simplest word at the very beginning of the dictionary has no precisely determined meaning. This word, as is well known, is the letter *a.* It is possible to distinguish the letter *a* when we restrict ourselves to a selected group of signs not specified in greater detail. On the other hand, it is easy to give examples of the letter *a* concerning which it might be impossible to decide whether they are in fact replicas of the letter *a* or of the letter *d* or *o.* This indeterminateness carries with it no serious practical consequences because it is always possible to disregard writing that is not clear. But this does not alter the fact that it is impossible to speak truly of the precise extension of a word.

The discussions of Aristotle concerning the arguments of the sophists seem to be of secondary importance. It is therefore a disquieting sign that there are still professional philosophers who take the Aristotelian method of definition *per genus et differentiam specificam*[2] seriously. But to-day even a moderately intelligent man recognizes that there are no words which have an absolute priority over all the others and that defined words as well as those in terms of which they are defined therefore have the same status. Both types of words were acquired in childhood as a consequence of habits formed in connection with observed objects, or simply as meaningless sounds to which we have finally become accustomed. But these are banal truths already known by Roscellinus, the Canon of Besançon, who lived during the Middle Ages when the halls of the royal palace were

[1] Cf. X. Pawlicki: *Historja filozofji greckiej* (*The History of Greek Philosophy*), Kraków, 1890, p. 267.

[2] Cf. Aristotle: *Metaphysica,* translated by W. D. Ross, Oxford, 1908, book vii (Z), ch. 12, 1037*b*.

lighted by pine torches and when the mode of travel was by imaginary magic carpets rather than aeroplanes. But the Scholastic tradition has a stronger hold on the minds of philosophers than the natural and mathematical sciences and social revolutions. The thought of Aristotle dominates Scholastic minds as if nothing had changed since his day.

Students who are influenced by the Scholastic mode of thought regard their verbal definitions as if they had been created by the mind of God, even though they frequently have too much good sense to take the matter very seriously. Unfortunately by pursuing this kind of activity they acquire a disdain for logic and become plastic material for irresponsible anti-rational doctrines.

It is usually said that a definition gives the so-called *genus et differentia specifica* of a term, but it is not generally observed that the attainment of a complete classification in conformity with this concept of definition will always remain a *pium desiderium*.

If a child sees a sparrow and asks what it is, he will be told that it is a sparrow. The sensible child will be satisfied with this answer when he hears the word *sparrow* for the first time because he feels instinctively that nothing else is involved. The word sparrow is a label which is attached to certain objects and nothing more is to be said. If this child now sees a canary he will say that it is a sparrow. When he is told that it is not a sparrow but a canary he will ask why this is the case. He will perhaps be answered because the canary has a yellow breast while the sparrow is grey. The child now knows the specific difference between canaries and sparrows but not the *specific difference of canaries*. Even a very accurate and natural description which is supplemented by photographs or multicoloured slides cannot show him the specific difference of a canary. Such a description is never sufficient. Ultimately it is necessary to point to a living model.

The same phenomenon occurs in all descriptive sciences, especially in grammar. A rule which is not accompanied by examples is of no value and the understanding of a rule increases as the number of examples increases. If many examples are given, it is even possible to dispense with the rules.

4. The philosophy of the late Edmund Husserl developed from his stubborn belief in the absolute character of the external structure of every-day language. This philosophy exerted a very strong influence upon German culture. It awakened the passion for seeking absolute criteria in all realms of human

thought, including civil law. The effects of phenomenology are very strong to-day. I have often had occasion to combat them. I fear that they have helped strengthen the spirit of intolerance and have weakened the capacity for thorough criticism.

Husserl believed in the existence of apriori laws which separate sense from nonsense. He wrote :

> " These laws which rule in the domain of meanings and serve to distinguish *sense* from *nonsense* are not the laws of logic in the fullest sense ; they provide logic with the *possible forms of meaning*, i.e. the apriori forms of complex significant meanings of '*formal truth*' and '*objectivity*' respectively, which then govern the 'logical laws' in the fullest sense. While the former laws guard against *nonsense*, the latter guard against formal or analytic contradiction, i.e. formal absurdity." [1]

This passage brings out the important truth which has been known to mathematicians for a long time, that the problems of logic cannot be confined to that of truth and falsity alone. The study of logic must begin with the problem of meaning. Unfortunately, however, Husserl combines the fundamentally false supposition that there are apriori laws of sense and nonsense with this truth.

In every-day language the transition between sense and nonsense is not sharp.

There exist various degrees of sense, ranging from the theorems of elementary arithmetic to metaphysics in the style of Heidegger and dadaistic poetry. Investigations of these phenomena are worth considering if they do not presuppose the existence of apriori laws of sense.[2] If the matter is understood in the same way as it was understood by Husserl, immutable and determinate forms must be sought where actually there are none, and this would spoil the whole undertaking.

Husserl was of the opinion that making investigations concerning linguistic phenomena depend upon psychology makes penetration into the essence of things impossible.

> " Modern grammar believes it must be based exclusively upon psychology and the other empirical sciences. On the other hand

[1] Edmund Husserl : *Logische Untersuchungen*, Bd. ii, Teil 1, Halle, 1913, pp. 294–5.

[2] Cf. Rudolf Carnap : "Überwindung der Metaphysik durch logische Analyse der Sprache," *Erkenntnis*, Bd. 2, 1931, pp. 219–241.

Cf. likewise the article of Professor Kasimir Ajdukiewicz entitled " Sprache und Sinn," *Erkenntnis*, Bd. 4, 1934, pp 100–138.

Cf. A. F. Bentley : *The Linguistic Analysis of Mathematics*, Bloomington, Indiana, 1932, a book which contains an interesting classification of the different degrees of meaning in mathematics.

the view is advanced here that the ancient idea of a *universal* and in particular of an *apriori grammar* obtains by our proof an indubitable foundation for the possible forms of meaning of the determining laws, and in any case determines *a* delimited sphere of validity." [1]

I have intentionally interpreted certain paragraphs of the original text, although it may be difficult to decide whether this has been done correctly. I have given an interpretation of this passage because I would like the reader to be able to orientate himself in some degree and because Husserl's conception is so vague and confused.

In the first place in order to conduct apriori investigations an appeal must be made to some system of thought which can rightfully be regarded as correct. If every-day language is to be subjected to criticism it is necessary to have at our disposal some other language which is better and which can be relied upon with complete confidence. However, the language of Husserl does not awaken complete confidence. Not only is it not better than every-day language but it is considerably worse. His language is unusually difficult to understand. It contains new linguistic creations, quotation marks, sentences which are much too long, parentheses, marks of notation, etc. It is difficult to discover the principle which permits this awkward mixture of triviality and subtle distinctions to be regarded as scientific language. There are indeed people who can intuit these things after a certain length of time but I fear that it is possible to intuit the most profound nonsense, as it is evident from the history of mythology.

As a result of his doctrine of apriorism, Husserl ascribes to expressions *intentional correlates,* i.e. certain fictional objects which are independent of us.

This is obviously a return to medieval realism. The doctrine of intentional correlates was developed in great detail but led to no concrete results. The belief in intentional correlates is fundamental to Husserl's views and any one who believes in this doctrine is subject to tragic despair and hopeless struggles with himself.

An interesting example of such an inner struggle may be found in a book devoted to a consideration of literature by Professor Ingarden, a well-known disciple of Husserl. In this book Ingarden tries to show that a literary work is the so-called

[1] Husserl: *l.c.*, p. 295.

heteronomical object which is independent of individual concretions.

Professor Ingarden writes as follows :

> "And if one spoke only of 'intentional correlates' and not simply of 'representations' as the psychologistic approach does, would this not be a mere quarrel over words ? If the concept here developed of literature is more sensitive and subtle and if talking of 'representations' is still very crude and primitive, does it not amount to the same thing in the end. . . ? But can literature really be reduced to a manifold of concretions ? Do not the numerous differences which we have exhibited between the work of literature and its concretions bear witness to the contrary ? The retort might be given : These differences exist only when one initially apprehends the Idea of the identical literary work which is manifested in its concretions as actually happened in our previous considerations. But what guarantees to us the identity of the literary product in contrast to its concretions especially if one concedes that the individual concretions differ from each other considerably and that the reader very often makes absolute the concretions which happened to be given to him and believes that he grasps in them the literary product itself ? and what in particular guarantees the identity of the literary work if it is read by different readers, i.e. what guarantees its *intersubjective* identity ? and what is identity in this case ? Would it not perhaps be most correct to say when all of us read one and the same literary product that although similar 'concretions' grow out of individual readings there is only one genuine illusion or error ? and finally : if the literary work is only an image of subjective operations which do not exist substantially (seinsautonom) the question arises how the literary creation exists when it is being read by no one." [1]

The only argument employed by Ingarden in dealing with these questions is that if the intersubjective meaning of propositions were abandoned there would be no common ideal science for all men. He still deludes himself that this meaning of propositions which is common to all men can be brought to the light of day through phenomenological investigations although he himself encounters many difficulties. New investigations which would have to be both extensive and difficult would be necessary to discover this common meaning. It would also be necessary to write a new book. Ingarden does not mention these difficulties but follows another method. He assumes the substantial existence of ideal concepts and bases his heteronomical objects upon them. I explicitly say he *assumes* because

[1] Roman Ingarden : *Das literarische Kunstwerk*, Halle, 1931, p. 371.

coming from a phenomenologist these words seem very paradoxical. They lead to the overthrow of this faithful disciple of Husserl, but at the same time they are the exultant cry of productive creative philosophy.[1]

Karl Bühler, the German linguist, follows in the footsteps of Husserl. While he abandons eternal and immutable processes, he fully believes in an intersubjective world. He complains that contemporary linguists, in particular the famous de Saussure, do not share his opinion.

Bühler writes:

> "The peculiar standpoint of De Saussure with regard to abstractions and generalizations and a similar opposition (*horror abstracti*) to my thesis of the ideality of linguistic forms among contemporary philologists whom I respect, is not very intelligible to me."[2]

He also appeals to the argument that the political economist who writes about the dollar or the zoologist who writes about the species of animals are also dealing with abstractions. But the point is precisely that the political economist who writes about the abstract dollar is under the illusion that it is a real existent and he cannot understand that the dollar is a plague which must be eradicated at all cost. Similarly, the zoologist who takes seriously the species about which he is writing will stubbornly oppose Darwin's theory. Even if he should accept this theory he will make advances only with great difficulty.

5. The discussion concerning every-day language centres about the problem whether the difficulties raised in connection with it can be eliminated through the use of the analytic method, i.e. through an explanation of them and the classification of the material at hand, or whether a complete reconstruction is necessary. Undoubtedly such a necessity follows from the existence of paradoxes which arise because this language permits the individual to define words by means of words, to speak about the fact that he is speaking, to think about the fact that he is thinking, etc., without any restrictions.

Logical paradoxes must be distinguished from semantical paradoxes.

The traditional Epimenides antinomy does not appear unless a proposition is affirmed. It is therefore neither a logical nor a semantical antinomy. Rather it should be called a dialectical

[1] Cf. *T. V. M.*, p. 65.

[2] Karl Bühler: "Axiomatik der Sprachwissenschaften," *Kantstudien*, Bd. 38, 1933, p. 56.

antinomy. It should be noted that it is not a formal antinomy, although it involves the vicious circle fallacy.

Epimenides the Cretan affirms that all Cretans are liars, i.e. that the Cretans always affirm false propositions. It is clear that this statement of Epimenides could never be true if it were assumed that it falls within its own scope. A proof that some Cretans are not liars can be given by employing the simple statement of Epimenides, which is a lie.

Since this conclusion is based upon the vicious circle fallacy, it must be agreed that the statement of Epimenides cannot fall within its own scope. This very important remark was first enunciated by Poincaré.[1]

Russell[2] reformulated the statement of Epimenides as the simple statement "I am lying." He interprets the latter statement as "There is a proposition which I am affirming and which is false" and obtains a formal antinony. This interpretation of the paradox can be explained in the same way as the original Epimenides antinomy.

It might be assumed that the proposition "I am lying" falls within its own scope. Such an interpretation of this proposition is not correct. No correct statement can be made about a proposition E unless the idea of a propositional function $F(X)$, whose value $F(E)$ is identical with the original statement, is abandoned. It is clear that E must be different from $F(E)$. Consequently no proposition can ever fall directly within its own scope.

This paradox is very instructive because it undoubtedly shows that the phrase: *all propositions* like the phrases *all properties, all expressions, all classes,* etc., is not clear and therefore leads to paradoxes. These paradoxes were of great importance in the history of logic because they led to the famous theory of logical types of Bertrand Russell. Later the opportunity to study the various kinds of paradoxes will present itself. Here I will consider only the paradox of Grelling which is of interest because like the paradox of Eubulides it occurs only in everyday language. It is much more suggestive because it does not depend upon facts but is purely formal in character.

A word which defines a property that belongs to itself will be called *autological.* The word *short,* for example, is autological

[1] Cf. Henri Poincaré: "The Latest Efforts of the Logicians," *The Foundations of Science,* translated by George Bruce Halsted, New York, 1929, sections vii and ix.

[2] Bertrand Russell and Whitehead: *Principia Mathematica,* 1st ed. Cambridge, 1910, vol. i, p. 62.

because it is short and thus defines a property which belongs to itself. Words which do not have the property they define will be called *heterological*. The word *long*, for example, is heterological because it defines a property which it does not have, since it is not long. It may now be asked whether the word *heterological* is autological or heterological. If it is autological it has a property which it does not define and is therefore heterological. But if it is heterological, it does not have the property it defines and therefore is not heterological but autological.[1]

These paradoxes decided the fate of rational metaphysics, because it turned out that if reasonings which lead to paradoxes are avoided, no progress can be made in metaphysics. The essence of metaphysics depends upon the fact that everything, all one's thoughts, all possible thoughts, all properties, all objects, etc., are being spoken about at once.

If the principle of contradiction is accepted this sort of reflection must be rejected as meaningless. In particular an individual cannot talk about what he is saying at the very moment he is saying it. It therefore follows, as has long been known by the philosophers of India, that the subject cannot be rationally investigated. An individual can speak about the subject *ex post*, i.e. the moment it becomes an object but not at the moment he is this subject. In a word the present time can never be the object of an individual's investigation. Not until later can the object of investigation be distinguished from the means by which the investigation was conducted. But at this time a new investigating subject appears which cannot be investigated objectively.

It follows from these considerations that the principle of contradiction does not permit complete knowledge, i.e. knowledge which includes the answer to all questions. The attempt to secure such knowledge will sooner or later conflict with sound reason.

6 A solution of the problem is given by the doctrines of the positivists on the one hand, and by dialectical materialism on the other. Both these doctrines reject the existence of objects which differ fundamentally from those encountered in experience. From the standpoint of methodology the difference between them lies in the fact that the positivists fixed considerably narrower limits for knowledge which is based upon sound reason and interpreted the suppositions of the natural sciences

[1] Cf. Hermann Weyl: *Das Kontinuum*, Leipzig, 1918, p. 2.

as hypotheses, while the materialists accept these suppositions without restrictions.

Both these doctrines developed from a common stem which goes as far back as the perpetually changing stream of existence of Heraclitus. The struggle of the sophists on the one hand and of the atomists on the other against the Greek idealists was their cradle. During the Middle Ages these doctrines lost their separate identity and their joint doctrine was called *nominalism.*

The doctrines of the nominalists completely overthrew idealism. These doctrines depend upon the complete elimination of such objects as concepts and propositions. It is affirmed that only certain sounds (*flatus vocis*) to which can be added certain expressions, i.e. certain combinations of letters, are involved. In addition to the previously mentioned Roscellinus, the representatives of medieval nominalism during the fourteenth century were the Dominican William Durand de St. Pourçain and the Franciscans Petrus Aureoli and William of Occam. Occam's Razor : *Entities should not be unnecessarily multiplied.* (*Entia non sunt multiplicanda praeter necessitam*) is important even to-day, and is employed as an argument against the claims made by the idealists.

It is obvious that the nominalistic position is compatible with the theses of both positivism and materialism because everything is reduced to what at first sight seems unattainable, i.e. sense data.

I think that the truth of nominalism follows irrefutably from our earlier discussion concerning every-day language. Probably nothing can make an idealist abandon his belief in the existence of ideal objects, but without question in practice only words and the automatic reactions evoked by these words are involved. This is hardly the perfection postulated by the idealists. These facts can be confirmed experimentally and consequently are indubitable. They constitute a foundation sufficient for constructing a theory of knowledge based upon sound reason.

David Hume was the first thinker to propose that sound reason rather than metaphysical criteria be regarded as the exclusive source of knowledge.

In contrast to the Greeks who saw no intermediate path between truth and complete agnosticism, Hume followed the path of compromise. He was convinced that it is better to abandon great pretensions than to be governed by a wild imagination. Because of his mild and resigned temperament he was able to avoid extreme scepticism but he was therefore unable to oppose the tendencies of his epoch. Consequently while he attained great success during his life, he had no great influence upon the important social movement which was to develop during the post-revolutionary period.

Hume was the type of sybarite who was well liked in elegant society and knew how to win its regard. His life was governed by the motto of Aristippus : *Only the present moment is ours.*[1]

Because of his mild temperament, Hume made no dangerous enemies in the course of his life. He wrote :

> " And though I wantonly exposed myself to the rage of both civil and religious factions, they seemed to be disarmed in my behalf of their wonted fury." [2]

Furthermore his social success was very great. Natanson writes as follows on this subject :

> " When Hume arrived in France he was received with the adoration due the royal family ; the noblemen idolized the great philosopher, the ladies implored him to join them in their boxes at the theatre ; *le gros David Hume* was like a sultan surrounded by charming slave girls in *tableaux.*" [3]

It can easily be understood that such an atmosphere influenced Hume's work. He completely ignored the creative power of ideas and reduced them to weak reflections of sense impressions. He did not dare dream that they can lead to an alteration of one's view of the world and ultimately transform life on earth. Thus his philosophy was a very narrow fragment and left much scope for romantic anti-rationalism.

Hume's relation to the mathematicians was especially unfortunate. His criticism of the infinitesimal calculus of the time was indeed justified but was too radical for that day and prevented the further development of the infinitesimal calculus. Its effect was, however, counteracted by the creative power of the calculus itself.[4] Thus Hume prepared the ground for Kant's apriorism and Hegel's anti-rationalism.

Hume's position was extremely individualistic. August Comte, the creator of the *positivistic* view of the world, applied

[1] Cf. Pawlicki : *l.c.*, p. 425.

[2] David Hume : " My Own Life," *An Enquiry concerning Human Understanding*, Chicago, 1930, p. xvi.

[3] Władysław Natanson : *Porządek natury* (*The Order of Nature*), Kraków, 1928, p. 201.

[4] Cf. David Hume : *A Treatise of Human Nature*, Oxford, 1896, book i, part ii, section iv.

it to the realm of society.[1] Positivism is the apotheosis of sound reason. It involves the complete rejection of metaphysics and of irrational dialectic. Its proponents were the first to attempt to formulate the aims of science and of social life in conformity with the principles of sound reason.

Comte's conception characterizes the scientist as one who can predict facts and make progress toward intellectual self-perfection or evolution. This view was a significant advance over the metaphysical and irrational fictions which prevailed at the time. Yet even in Comte's doctrine too much compromise was involved. He did not observe the very important law, that science and all life on earth do not develop in accordance with law but through the supplementation and perfection of that which has already been. Even the power to predict without conscious effort what will happen does not suffice. It is also necessary to be able to predict that if such an apparatus is constructed, with its help it will be possible to deal with that which threatens. In other words it must be possible to construct, it must be possible to create. The construction of new systems of thought, new criteria of beauty, and new forms of life is the essence of progress in science, art, and life. Creative power is not the result of slow evolution. There has been no continuous transition from the wings of Daedalus to aeroplanes. The discovery of radium and Roentgen rays and the invention of radio were as sudden and unexpected events as revolutions. Beginning with Byzantine and Gothic art and ending in the impressionistic art of the present day, the development of art has been characterized by a series of revolutions. Life on earth has developed in a similar fashion. It is a well-known fact that the path of progress is marked by pools of blood and unconditional revolutions.

Everything which has been brought about through evolution might have been accomplished elsewhere and much earlier through revolution, but in any case it would inevitably have been accomplished. This important truth was discovered by Karl Marx, the creator of the theory of historical materialism. Comte's successors disregarded this discovery. Although the adherents of positivism were very great thinkers, they have had no significant influence upon the history of culture in recent times.

Ernst Mach, one of the most vehement critics of the

[1] Cf. Überweg-Heinze: *Grundriss der Geschichte der Philosophie*, Bd. 9 Aufl., Berlin, 1902, p. 366 ff.

metaphysical deceptions in contemporary science, was undoubtedly the greatest of Comte's successors.

One of Mach's greatest merits was that he did not hesitate to reduce mathematics to the same level of triviality and banality as communication and trade. However, his views had no immediate influence upon the development of contemporary mathematics, which under the influence of Georg Cantor was extremely idealistic in character. But recent work, which will be considered later, completely corroborates his opinions.

Mach wrote:

> "Numbers are often characterized as 'free creations of the human mind'. The admiration for the human mind expressed here is very naturally opposed to the complete and imposing edifice of arithmetic. However, much more is required for the understanding of these creations if their *instinctive beginnings* and the circumstances which engendered the need of them are considered." [1]

Mach saw no essential difference between the most sublime mathematical constructions of our day and the calculations of children and primitive peoples. Mach was right. But it must be kept in mind that he lacked the arguments necessary to support this thesis. Such arguments can only be furnished by detailed investigation of all the intervening stages in the development of mathematics, beginning with these economic calculations and ending with the system of rational metamathematics. Likewise is required the clear conviction that only the consistent performance of the elementary operations necessary in performing economic calculations and suppositions which formulate precisely the results of such operations, are involved. This matter will be considered later. Here I only wish to observe that this economic point of departure must be justified. The reader has the right to expect me to do more than merely make promises on this score and to employ some theoretical principles. However, I am sure that I will be able to fulfil my obligations.

It should be pointed out that the late Professor Moritz Schlick was one of the philosophers of our day who was aware that there is no essential difference between scientific knowledge and the knowledge of daily life. He observed that it is not proper to apply the concept of probability to measurements which are made within wide limits. For example the assertion that the distance between one's home and the university is greater than

[1] Ernst Mach: *Erkenntnis und Irrtum*, 5. Aufl., Leipzig, 1926, p. 327.

10 *cm.* is a true proposition.[1] If a proposition of this type were regarded as dubious, it would be impossible to define the concept of probability because there would be nothing upon which this concept might depend. I think that this remark is very important for the understanding of the role of the elementary criteria of sound reason. These criteria give the greatest amount of certainty which can be obtained and that is why it is believed that the propositions derived by their use are true.

Nevertheless Schlick succumbed to the illusion that there are propositions which define a certain state of affairs unequivocally. He called such propositions true *ex definitione.*[2]

7. The struggle concerning a view of the world which is based upon the criteria of sound reason was reduced by the eighteenth century materialists to a critique of Cartesianism on the one hand and of Berkeleyanism and the sensualism of Locke and Condillac on the other.

The prejudice concerning the substantiality of the soul led Descartes [3] to the paradoxical assertion that in contrast to people animals are machines. But it is clear that there is no essential difference between men and animals. If animals are machines, it immediately follows that men are likewise machines. De la Mettrie, who was born in 1709, was the first thinker who dared enunciate this simple truth publicly.[4]

Diderot, one of the authors of the *Encyclopædia,* openly mocked Berkeley :

> " In a moment of madness, the sentient piano imagined that it was the only piano existing in the world, and that the entire harmony of the universe was accomplished within itself." [5]

The conception of sound reason was advanced very far by the activity of these thinkers. The sound reason of Socrates was pure mythology compared with this new conception.

The second period of development of thorough criticsim, based upon a materialistic world view, began when *Das Kapital,* by Karl Marx, was published in 1867.

Because of the historical connection between dialectical

[1] Moritz Schlick : *Allgemeine Erkenntnislehre,* 2 Aufl., Berlin, 1925, p. 133.
[2] Schlick : *l.c.,* p. 55.
[3] Renée Descartes : " Discours sur la méthode," *Œuvres,* nouvelle ed. par J. Simon, p. 37.
[4] Cf. Julien Offray de la Mettrie : *Man, a Machine,* 1785, translated by Gertrude C. Bussey, and revised by Mary W. Calkins, Chicago, 1912, pp. 89, 148.
[5] Denis Diderot : *Œuvres Complètes,* Paris, 1875, vol. ii, p. 118. Cf. Nikolai Bukharin : *Historical Materialism,* authorized translation from the 3rd Russian edition, New York, 1925, p. 58.

materialism and Hegel's anti-rational dialectic and because the very name dialectical materialism has many anti-rational associations, this doctrine is generally regarded with a certain prejudice. Actually, as has already been emphasized, it is not only the negation of anti-rationalism but it is a doctrine which is based upon the principles of sound reason entirely. Its strength is its complete opposition to idealistic metaphysics and its introduction of the constructive method into sociology. Marx's reduction of the concept of economic value to the concept of human labour has nothing in common with the aims of anti-rationalism. On the contrary it is an example of scientific precision and uncommon simplicity.

Bukharin wrote the following concerning the dialectic of Marx:

> " For Marx, dialectics means evolution by means of contradictions, particularly a law of ' being ', a law of the movement of matter, a law of motion in nature and society. It finds its expression in the process of thought. It is necessary to use, the dialectic method, the dialectic mode of thought, because the dialectics of nature may thus be grasped." [1]

This definition seems to me to be unfortunate. No special method is necessary to understand the changes which occur in nature and in particular to understand the fact that all disturbances in nature tend to a state of equilibrium. Hegel thought that motion can be understood only in terms of moving thought. This is plainly absurd. The prejudice which results from this unfortunate conception is the source of misunderstanding. Sound reason and the constructive method which among other things lead to the establishment of equilibrium with regard to motion are sufficient to understand the changes which occur in nature.

These means are sufficient to understand changes in nature of the type with which Marx and his successors were concerned. Only if an intuition of the essence of movement in the sense of Bergson [2] were desired, would it be possible to complain with him of the ineptness of the constructive method. But the representatives of Marxism did not have such lofty ambitions. As a result, when they talk about the dialectical method they manifest a confusion of concepts which is entirely unnecessary. But

[1] Cf. Nikolai Bukharin: *l.c.*, p. 75

[2] Cf. Henri Bergson: *Matter and Memory*, translated by Nancy M. Paul and W. Scott Palmer, London, 1911, pp. 290–1, 321.

Marx himself observed the dangers which are involved in Hegel's dialectic. He wrote on this subject :

> "In its mystified form, dialectic became the fashion in Germany, because it seemed to elucidate the existing state of affairs. In its rational form, it is a scandal and an abomination to the bourgeoisie and its doctrinaire spokesmen, because, while supplying a positive understanding of the existing state of things, it at the same time furnishes an understanding of the negation of that state of things, and enables us to recognize that that state of things will inevitably break up ; it is an abomination to them because it regards every historically developed social form as in fluid movement, as transient ; because it lets nothing overawe it, but is in its very nature critical and revolutionary." [1]

The fundamental thesis of Marxism is mutability. According to Engels it depends upon the conception of the world as a process, i.e. as something which continually develops.[2] This thesis must be accepted to-day by all philosophers because it is based upon unquestionable natural and historical facts. If it were desired that this thesis be rejected, troublesome contradictions would result or at best artifical subterfuges would have to be employed. Such would likewise be the case if the second fundamental thesis of Marxism, namely the close dependence of intellectual life upon physiological processes were rejected. Confirmation of the fact that it is possible to forget one's mother tongue following injury to certain regions of the brain, although a foreign language which was imperfectly known is still remembered, makes it difficult to doubt that this dependence is complete.

Thus the establishment of these theses is based only upon the results of investigations in the exact sciences and there is no need to employ special methods of reasoning to establish them. It suffices to keep in mind that the realm of applicability of concepts is always restricted. Bukharin pointed this out when he wrote :

> "Hence our distinctions, as made above ; they hold good—as we have said—when understood *dialectically*, i.e. within certain bounds, conditionally, according to circumstances." [3]

It is clear that if this were all that is meant by dialectical materialism, all discussion would be superfluous.

[1] Cf. G. Plekhanov : *l.c.*, pp. 30–1.
[2] Cf. A. Deborin : *Wwedenie w filosofiu dialekticzeskogo materializma*, Moskwa-Leningrad, 1931, p. 205.
[3] Bukharin : *l.c.*, p. 85. (Author's italics.)

Actually the Marxists keep their reasonings within these bounds. For example the works of Bukharin, one of the foremost contemporary Marxists, at no point go beyond the bounds of sound reason. It may even be said that they are written unusually clearly and in this respect might be regarded as a model for western philosophers. The only weak point in the work of the Marxists is connected with their discussion of the Hegelian tradition and is difficult to explain. The moment they begin to speak about Hegel, confusion of concepts appears.[1] They speak of Hegel as earnestly and in the same style as they speak of other matters and in one breath they cite improbable Hegelian nonsense and very important scientific theories. The impression given is either that the author has fallen into momentary error or that he is laughing at the reader. It may be said in defence of the Marxists only that this same weakness may be found among professional logicians.

It may also be observed that this defect is a secondary result of the persecutions undergone by the Marxists in pre-war years. When people who in general are guided by sound reason, cannot or do not wish to accept certain truths which are obvious to us, we tend to have an irrational reaction. I observed such a tendency in myself when almost everyone I knew tried to dissuade me from going on with logic, because it is a fruitless plaything. My position was even more difficult because the few professional logicians of the time were proponents of the formal approach to logic, a method which was foreign to me. Who knows what would have happened if this situation had not changed ?

I will now quote the following passage from Bukharin in order that the reader may familiarize himself with the attitude of the Marxists toward Hegel's dialectic. He wrote :

> " We have now to consider the final phase of the dialectic method, namely, the theory of sudden changes. No doubt it is a widespread nothing that ' nature makes no sudden jumps ' (*natura non facit saltus*). This wise saying is often applied in order to demonstrate ' irrefutably ' the impossibility of revolution, although revolutions have a habit of occurring in spite of the moderation of our friends the professors." [2]

It is difficult to imagine more convincing reasoning. But why is this method dialectical when it is clear that simple reference to facts is involved. Bukharin, however, was not satisfied by this. It is possible that just because he wished to give the

[1] Bukharin : *l.c.*, p. 75. [2] Bukharin : *l.c.*, p. 79.

reader the impression that some extraordinary method is necessary, he asked whether nature is really as temperate as it is said to be.

After the above discussion this question might seem superfluous, but it would do no harm to keep in mind such cosmic catastrophes as the diremption of satellites, volcanic eruptions, avalanches, etc., and *last but not least* quantum physics.

Instead of this a quotation from Hegel's *Science of Logic* is found which contains the following sentence :

> " Yet we have seen cases in which the alternation of existence (of that which is, which exists, *des Seins*) involves not only a transition from one proportion to another, but also a transition, by a sudden leap, into a *quantitatively*, and, on the other hand, also *qualitatively* different thing (*Anderswerden*) ; an interruption of the gradual process (*ein Abbrechen des Allmählichen*), differing qualitatively from the preceding, the former, state." [1]

If it is supposed that Bukharin understood these words I could never take him seriously. I prefer to think that he quoted them because of a certain snobbery and because he wished to astound the reader.

Dialectic appears to even less advantage in the writings of Thalheimer. He openly comes forward as an opponent of logic, which he identifies with the doctrine of Aristotle on the basis of the Hegelian dialectic. Thalheimer holds forth in phrases worthy of Hegel and attacks the principle of identity $A = A$ on the ground that movement is continuous as Heraclitus had maintained.

I suppose that the Chinese students to whom Thalheimer lectured in Moscow, being in a desperate state of mind, were inevitably influenced by his views.

There is a simple answer to this type of argument: Heraclitean changes in the letter A are not sufficiently strong to affect our relation to this letter. Consequently it is utterly pedantic to take into account changes of this kind, which by their very nature necessarily lead to paradoxes. There are obviously other kinds of changes which may cause serious difficulties. For example another letter might be substituted surreptitiously for the letter A and it is impossible to be absolutely sure that this has not been done. But this state of affairs will distress only those with excessive pretensions who are not content with ordinary human certainty but strive for absolute and therefore

[1] Bukharin : *l.c.*, p. 80. (Italics Bukharin's.)

superhuman certainty at any cost. There is no remedy for pretensions of this kind. The situation is similar to the case of a person who does not wish to ride on a train because an accident might occur and who therefore makes his journeys on foot. Such problems will not be considered here.

It follows from these considerations that dialectical materialism like positivism was caught on a shoal which rises from the weaknesses at the bases of the exact sciences. If these sciences are regarded as complete in spite of their omissions and in spite of the presence of idealistic and metaphysical elements in them, progress becomes impossible. Even the keenest minds and minds which oppose the aims of the anti-rationalists cannot escape the chaos of the problems which present themselves if work is not begun at the very bottom and if the sources of the simplest human knowledge, e.g. the knowledge required to learn multiplication tables, are not examined. Whether it is desired that life on earth be elevated to heights unknown, or simply that the truth be obtained and impatient curiosity be satisfied, ant-like, it is necessary to examine the foundations of science, even those which are most elementary. Considerable confusion of concepts and very marked differences of opinion will be found concerning these foundations. Progress will be made only at the cost of great effort and stubborn patience. If any progress is actually made the results of these efforts will recompense us a hundredfold. Self-confidence and confidence in the bright future of humanity will be developed and independence of the welter of confused concepts about us will be obtained. I think that these values are worthy of great sacrifice.

Chapter III

THE DEVELOPMENT OF THE CONCEPT OF NUMBER

1. As is well known it is very easy to teach a child to count with pebbles or apples. However, difficulties arise when one attempts to introduce pure numbers. This is evident if one but considers the history of human thought. The arithmetic of the Egyptians and the Babylonians was confined to the sphere of practical applications and therefore was entirely clear. Difficulties first arose when the Greeks created the concept of natural number. This very important step was taken in a manner which is hard to describe ; nevertheless this process was certainly far less simple and clear than the related processes in the arithmetic of natural numbers. The arithmetic of the Greeks was part of metaphysics and never was separated from the disturbing problem : how it is possible that natural numbers have independent existence ?

The Pythagoreans are known to have favoured the view that natural numbers are the only real substances and that the existence of all things may be reduced to them. They held that all things with the exception of the natural numbers have no independent existence and they regarded these things as relations between numbers. They associated with this conception of number a mystical cult and a feeling of awe toward the mystery of existence which may be found even to-day among the metaphysically inclined.

Pythagoras observed that the lengths of the strings corresponding to the four tones which he knew : *C*, *F*, *G*, *c*, are expressed as certain relations of the numbers 1, 2, 3, and 4. Pawlicki writes :

> ". . . seeing that in the decimal system, which he introduced into Greece from the East, all other numbers originate from the first ten through addition, he declared that all numbers are contained in the first decade. Furthermore because the first ten numbers or the first decade is the result of the addition of the first four numbers ($1 + 2 + 3 + 4 = 10$), which are the basis of musical harmonies, it seemed to him that the first decade and all other numbers which are possible, are contained in the 'first divine group of four' (τετρακτύς). Likewise all harmonies

arise from the first divine τετρακτύς. The latter will be the creator of all spiritual and material laws because these laws must also be harmonies." [1]

I have intentionally dwelt at length upon these speculations. It will be seen later that the speculative element, which originated at the same time as Greek mathematics, has survived up to our times and even to-day is very strong although it has been somewhat curbed. It is difficult to decide whether or not this element influenced the development of mathematical theorems ; however, it greatly hindered the construction of the foundations of mathematics.

However Nichomachus of Gerasa, the Pythagorean, who lived during the second century A.D., did not have the courage to derive all natural numbers from the number 1 by means of repeated additions. He was of the opinion that the number 2 as well as the number 1 has an exceptional position in the world of numbers, because the law :

$$a \times a > a + a$$

is satisfied by all natural numbers with the exception of 1 and 2.[2] For Nichomachus numbers were clearly fetishes of some kind.

The worship of natural numbers is evident at every step in the work of Nichomachus. Among other things he considered the theory of perfect, deficient, and abundant or super-abundant numbers.

Perfect numbers are equal to the sum of their factors. Numbers greater than such sums are called deficient ; numbers less than such sums are called abundant or super-abundant. [3]

For example, the number 6 is a perfect number, because

$$1 + 2 + 3 = 6\,;$$

the number 8 is a deficient number, because

$$1 + 2 + 4 < 8\,;$$

the number 12 is an abundant number, because

$$1 + 2 + 3 + 4 + 6 > 12.$$

It should be noted that the common factor 1 is invariably added to each sum of the factors. If it were omitted the perfect number 6 would immediately become a deficient number. Nichomachus did not observe this danger and employed

[1] X. Stefan Pawlicki. *Historja filozofji greckiej* (*The History of Greek Philosophy*), Kraków, 1890, p 200.

[2] Cf. Nichomachus of Gerasa : *Introduction to Arithmetic*, translated by Martin L. D'Ooge, University of Michigan Studies, Humanistic Series, vol. xvi, New York, 1926, p 117.

[3] Leonard E. Dickson : *History of the Theory of Numbers*, Washington, vol. i, 1919, p. 4.

metaphysical arguments to establish the importance of this classification. He wrote the following concerning deficient numbers :

> "It is, as if some animal should fall short of the natural number of limbs or parts, or as if a man should have but one eye, as in the poem, 'And one round orb was fixed in his brow'; or as though one should be one-handed or have fewer than five fingers on one hand, or lack a tongue, or some such member."[1]

Later he added the following words :

> "It comes about, that even as fair and excellent things are few and easily enumerated, while ugly and evil ones are widespread, so also the superabundant and deficient numbers are found in great multitude and irregularly placed—for the method of their discovery is irregular—but the perfect numbers are easily enumerated and arranged with suitable order."[2]

These statements have been cited intentionally because, while nothing similar is to be found in mathematical treatises, closely related methods of appraisal may be met in conversation with prominent mathematicians, who as far as possible try not to go beyond inherited ideas.

The conceptions of perfect, deficient, and abundant numbers had a great influence upon medieval thinkers, and even today have not ceased to be important. In the ninth century Alcuin attributed the imperfection of the human race to the fact that in Noah's ark there were 8 souls and 8 is a deficient number.[3]

In addition to deficient, perfect, and abundant numbers there are likewise *amicable* numbers, i.e. numbers such that each equals the sum of the aliquot divisors of the other. The numbers 284 and 220 are examples of amicable numbers.

Amicable numbers are important even to-day, especially in America where Dickson has been concerned with them. He has even invented a new kind of amicable number called amicable triples.

2. Investigations concerning the essence of natural numbers have made the matter of numerical symbolism seem unreal. The Greeks had no simple numerical signs but denoted the numbers 1, 2, . . ., 9, 10, 20, . . ., 100 and 100, 200, . . ., 1000 by the letters of the old Greek alphabet.[4]

[1] Nichomachus of Gerasa : *l.c.*, p. 208.
[2] Nichomachus of Gerasa : *l.c.*, p. 209.
[3] Dickson : *l.c.*, p. 4.
[4] Cf. Frank E. Robbins and Louis C. Karpinski : "Studies in Greek Arithmetic," contained in *Nichomachus of Gerasa, l.c.*, p. 68.

They performed addition by writing the numbers side by side and then writing a stroke above the line. Multiplication was characterized by the sign $\times$, but was performed by writing one of the two numbers to be multiplied beneath the other and drawing a line beneath them both, just as addition is performed to-day. Fractions were represented in a very complicated way: The fraction $\frac{17}{25}$ was written $\iota\xi'\kappa\epsilon''\kappa\epsilon''$ where $\iota\xi'$ denotes 17 and $\kappa\epsilon''$ denotes 25. Sometimes the denominator was placed in the position occupied by an exponent to-day.

Operations on fractions were already known to the Babylonians. It may be that this method of notation was transmitted to Greece from Babylonia since there is evidence that the methods of the Babylonians influenced Greek science.[1]

The analogy between the arithmetic of natural numbers and the arithmetic of irrational numbers was not observed because, from the standpoint of metaphysics, natural numbers and fractions differ in character. Natural numbers are regarded as independent existences, while fractions are regarded as fictions. Karpinski contends[2] that the preponderance of metaphysical considerations in Greece was due to the fact that the Greeks had no well worked out system of notation.

The whole matter reduces to the fact that the concept of a fraction cannot be derived from that of a natural number. To define a fraction it is necessary to employ one of the following concepts: segment, class, relation, or expression. But in each of these cases the natural numbers cease to have an exceptional role and the Greeks wished to avoid this consequence at all cost.

In an interesting discussion, Heinrich Scholz[3] noted that the essential reason why the Greeks did not develop a concept of irrational numbers was that they had no concept of rational numbers. In support of this theses Scholz cited a passage from Plato's *Republic* from which it is clear that the Greeks regarded fractions as paradoxical constructions. They were of the opinion that units cannot be subdivided and therefore did not believe in the existence of fractions. They tolerated operations on fractions as purely practical activities with no scientific value. Consequently it is not suprising that fractions were attributed to the world of illusion and disdainfully thrust aside.

[1] Cf. Otto Neugebauer "Zur vorgrieschischen Mathematik," *Erkenntnis*, Bd. 2, 1930, pp 122–134.

[2] Robbins and Karpinski: *l c.*, p 66

[3] Cf. Heinrich Scholz: "Warum haben die Griechen die Irrationalzahlen nicht aufgebaut?", *Kantstudien*, Bd. 33, Berlin, 1928, p 65.

This view was also supported by the fact that the theory of geometric proportions created by Plato's contemporary Eudoxus[1] made operations with fractions possible without introducing any special signs by means of a circumlocution. It is known that instead of writing: $a = 3\,b$, the auxiliary segment k can be used and an appeal made to the equalities $a = 3\,k$ and $b = 5\,k$. For geometrical purposes this method is sufficient.

It seems to follow from considerations in Aristotle's *Metaphysica* that Aristotle was on the road to emancipation from the prejudice concerning the primacy of natural numbers. Aristotle did not regard natural numbers as independent existences but sought independent units in the sphere of daily life. However, he encountered here the vagueness of popular concepts and became involved in dialectical investigations on this theme. And so Aristotle was concerned with the fact that:

> "... of things that are called one in virtue of their own nature some are so called because they are continuous.... Of these themselves, the continuous by nature are more one than the continuous by art. A thing is called continuous which has by its own nature one movement and cannot have any other."[2]

This appeal to the concept of a rigid body is but a step removed from reliance upon geometrical segments. Yet Aristotle was satisfied to make ontological investigations, although he did not draw the consequences which follow from them.

It should be observed that the prejudice concerning the primacy of natural numbers made very difficult the adjustments which became necessary when new kinds of numbers were created as mathematics developed. Young writes as follows concerning the discovery of negative numbers:

> "The first writer who appears to have recognized the existence of negative roots of a quadratic equation was the Hindu Bhaskara, in a work written about the year A.D. 1150. He gives $x = 50$ and $x = -5$ as the roots of $x^2 - 45\,x = 250$; 'but,' says he, 'the second value is in this case not to be taken, for it is inadequate; people do not approve of negative roots.' For centuries thereafter people did not approve of negative roots.

[1] Cf. Jan Sleszyński: "O pierwszych stadiach w rozwoju pojęć nieskończonościowych," *Poradnik dla Samouków* ("Concerning the first stages of the development of the concept of the infinitesimal," *Home Study Course*), tom. iii, Warszawa, 1923, p. 60.

[2] Aristotle: *Metaphysica*, translated by W. D Ross, Oxford, 1908, bk. v (Δ), ch. vi, 1015*b*, ll. 36–1016*a*, l. 6.

> The German mathematician Michael Stifel, speaks, in 1544, of numbers which are 'absurd', or 'fictitious, below zero', and which arise when 'real numbers above zero' are subtracted from zero."[1]

Even during the nineteenth century complex numbers were still regarded as fictional constructions. For example, the German psychologist Theodor Lipps placed them on the same plane as square circles and other contradictory objects.

3. The Pythagorean philosophers maintained that whole numbers are actual substances and real phenomena are relations between numbers. However, Nichomachus of Gerasa, who was influenced by Plato and Aristotle, admitted other categories than that of quantity. He therefore regarded numbers as

> "superior kind of forms, out of which the other forms are made and under which they are classified."[2]

This modification resulted from the great catastrophe which befell the doctrines of the Pythagoreans during the period when the school was still greatly influenced by the works of the master. The problem of the determination of the numerical relation between the side and diagonal of a square was the source of this catastrophe. This apparently simple and elementary problem led to entirely new and unexpected phenomena.

The investigations on this problem conducted by Pythagoras led to the discovery of incommensurable segments. In particular Pythagoras was convinced that the side and diagonal of a square are not commensurable. This theorem is generally known to-day. I will now give a proof of this theorem which seems to me to be unusually simple.

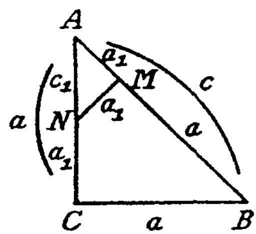

FIG 1.

If in a given square the length of the side a is marked off on the diagonal c and the remaining segment denoted by a_1, and if at the point M which divides the diagonal into two segments having the ratio $a : a_1$, the perpendicular to the diagonal is constructed, the right isosceles triangle $A\,M\,N$ will be obtained. It may easily be confirmed that N, the point of intersection of the perpendicular and the side $A\,C$, divides $A\,C$ into two segments having the ratio $a - a_1 : a_1$.

[1] John Wesley Young: *Lectures on the Fundamental Concepts of Algebra and Geometry*, New York, 1930, p. 107

[2] Robbins and Karpinski: *l.c.*, p. 97.

Let it now be supposed that there exists a certain segment j which is contained in the segment c, γ times without remainder, and in the segment a, α times without remainder, i.e. $a = \alpha j$, and $c = \gamma j$. Let the difference $\gamma - \alpha$ be denoted by α_1, the difference $\alpha - \alpha_1$ by γ_1 and the segment $a - a_1$ by c_1. It is clear then that

$$c_1 = \gamma_1 j \text{ and } a_1 = \alpha_1 j.$$

If then the side and the diagonal of a square have a common measure, there must exist a smaller square, whose side and diagonal have the same common measure. In this way is obtained an infinite sequence of ever decreasing squares whose side and diagonal invariably have the same measure. But it is clear that there can be at most $\gamma - 1$ segments less than c which are measurable by the same measure j and there cannot be infinitely many different squares with diagonals less than c which contain the segment j without remainder. The supposition which has been made is therefore false and it must be agreed that there is no segment which can be obtained without remainder in both the side and diagonal of a square.

Immediately upon its discovery this theorem made a strong impression upon the Pythagoreans. It was the first theorem ever met which contained an example of an infinite regress. This was the first time that a transgression of the bounds of finite arithmetic was ever observed.

Sleszyński writes as follows concerning this discovery :—

> "According to tradition although Pythagoras sacrificed one hundred bulls to the gods, he ordered his disciples to keep this discovery a secret because he regarded it as dangerous to his doctrine that everything must be a number." [1]

This phenomenon and similar manifestations of restraint in connection with infinite processes were the bases of the prejudice of the Greeks concerning finitism.

Sleszyński strongly opposed this prejudice. In a recent discussion with Spengler, Scholz [2] touched upon this matter.

Because of the importance of this whole discussion, this theme is worthy of further consideration.

The prejudice of the Greeks concerning finitism is a consequence of a simple misunderstanding. This prejudice goes so far, that even euclidean geometry which is based upon the

[1] Jan Sleszyński : *l.c.*, p. 59. [2] Heinrich Scholz : *l.c.*, p. 60.

concept of the infinitely long straight line is regarded as the product of finitistic thinking. Professor Schrödinger, the famous creator of wave mechanics, made such a statement in one of his popular lectures. He compared " the clear, transparent and rigid structure of euclidean geometry " with " the plain, simple and limited forms of the Grecian temple." [1]

It should be observed that in the Greek world, which was finite and one in which circular motion was regarded as the most perfect motion,[2] the discovery of the infinitely long straight line was difficult and required a special effort of the creative imagination. Minds which from childhood are accustomed to operating with the concept of parallel lines which never intersect, require the expenditure of a certain amount of effort to understand that what is concerned here is something not only in itself unintelligible but also something very hard to grasp intuitively. It should be kept in mind that the illustrations of parallel lines given in school, for example, perpendiculars, the rays of the sun, etc., are examples of lines whose point of intersection while very remote, actually exists. Examples of parallel lines cannot be given and consequently they must be regarded as fictions. Contemporary physicists who accept Riemannian geometry as well as Einstein's general theory of relativity, take this fact into account and do not admit the existence of infinite lines. If this fact is carefully considered the idea suggests itself that precisely the minds of present-day men rather than those of the Greeks are finitistic. Actually the Greeks, whose world was invariably closed, created the concept of the infinite line, while present-day physicists, who among other things must account for the phenomenon of the infinitely expanding world, have relied upon the concept of the finite straight line.

4. There was ecstatic admiration for the mysterious properties of natural numbers even in the days of the passing of Greek arithmetic. The rebirth of arithmetic was brought about through the introduction of purely formal accounts. It resulted from the collective efforts of the Arabs. (Al Battani, Al Biruni, Nasîr Eddîn, and others.) In the ninth century the Arabs began to study Greek mathematics and eagerly turned to the issues raised by it. They thrust metaphysical questions aside. They did not reflect upon the essence of natural numbers, fractions, or irrational

[1] Erwin Schrödinger: "Is Science a Fashion of the Times?", translated by Dr. James Murphy, *Science and the Human Temperament*, New York, 1935, p. 101.
[2] Cf. Aristotle: *Metaphysica*, *l.c.*, 1016*b*, ll. 16–17.

numbers, but created the present-day system of decimals and operated with sines and cosines. These purely formal achievements made the Arabs the real creators of algebra and trigonometry.[1] In the twelfth and thirteenth centuries their investigations influenced the famous Leonardo Fibonacci and in the fourteenth century Levi ben Gerson[2] who added important and orginal contributions to those of the Arabs. However, it was not until the end of the fifteenth century that Europe freed herself from the beliefs about natural numbers inherited from the Greeks through Boethius. At that time because of the development of commercial interests the problems of the theory of perfect, deficient, abundant, and amicable numbers became secondary. The first arithmetic in which these problems were omitted appeared in Venice in 1484. It was a commercial arithmetic written by Pietro Borghi.[3] The first trigonometry text was written by Johann Müller who was also known as Regiomontanus.[4]

The thinkers of the sixteenth century followed the formal procedures initiated by the Arabs. The solution of third degree equations achieved by Scipio Ferro [5] was a result obtained by calculations alone. It might even be called a pasigraphic result because it was of no practical value.

Systematic attempts to set up a system of numerical signs did not begin until the seventeenth century. Historians have mentioned three investigators who worked on this problem: Oughtred, Hérigone, and Leibniz.[6] The last mentioned became known as the co-creator of the infinitesimal calculus because he established an adequate symbolism for this branch of mathematics which has been retained up to the present day. Leibniz was an enthusiast for symbolism, and expected considerable progress in science to result from the use of symbolism. He wrote to L'Hôpital on April 28, 1693:

> "I dare say that this is the last effort of the human mind, (*characteristica generalis*), and, that when this project shall have been carried out, all that men will have to do will be to be happy, since they will have an instrument that will serve to exalt the intellect, not less than the telescope serves to perfect their vision." [7]

[1] Gino Loria: *Storia delle Matematiche*, Torino, 1929, p. 327 f.
[2] Loria: *l c.*, p. 422.
[3] Cf. David Eugene Smith: "The First Great Commercial Arithmetic," *Isis*, vol. 8 (1), 1926, pp. 41–9.
[4] Cf. Loria: *l c.*, p. 434.
[5] Cf. Loria: *l.c.*, vol. ii.
[6] Cf. Florio Cajori: "Leibniz—The Master—Builder of Mathematical Notation," *Isis*, vol. 7, 1925, p 412.
[7] Cf. Cajori: *l.c.*, pp. 417–18.

In the same letter, Leibniz defined his general characteristic as follows :

> " One of the secrets of analysis consists in the characteristic, that is, in the art of skillful employment of the available signs . . ." [1]

Hölder writes as follows on the views of Leibniz concerning natural numbers :

> " Leibniz does not say very clearly how he conceives number ; nevertheless because he designates the numerical formuli :
>
> $$2 = 1 + 1$$
> $$3 = 2 + 1$$
> $$4 = 3 + 1$$
>
> as the *definitions* of the numbers 2, 3, 4 . . ., it can well be supposed that he wishes to regard number as the place sign (of the decimal system of numbers [2]) and that he denotes an advance of one member in the series of place signs, as the addition of the number 1." [3]

The conception of Leibniz was ingenious but contained one weakness. The differential sign *dx* which he introduced concealed the absurd concept of the infinitesimal which threatened to lead to contradictions. During the eighteenth century criticism of this concept was begun by D'Alembert and at the beginning of the nineteenth century systematic work on its elimination from mathematics was undertaken by Cauchy and Abel. The discussion of these matters will be continued later. [4]

At this time, however, it need only be observed that this whole discussion led to the realization that it is not possible to create signs arbitrarily, because subsequent contradictions still remain a possibility. Consequently in the course of the nineteenth century there followed a return to the realistic conception of natural numbers.

However, in the second half of this century natural and psychological investigations on the one hand, and the failure of efforts to reduce real numbers to natural numbers on the other hand, prompted a reversion to the sign conception with natural experience as the guarantee against contradictions.

Helmholtz wrote :

> " I regard arithmetic or the study of whole numbers as a method based on purely psychological facts by means of which

[1] Cf. Cajori : *l.c.*, p. 417.
[2] Remark of the author.
[3] Otto Hölder : *Die mathematische methode*, Berlin, 1924, p. 163.
[4] Cf. 8.3. 8.4. and 8.7

is taught the correct use of a system of signs (namely of the numbers) which can be extended and improved without limit. In other words arithmetic investigates the different ways of combining these signs . . . which lead to the same final result." [1]

I think that Helmholtz discovered the essence of arithmetic. Frege in criticizing this passage, maintained that for Helmholtz signs aquired a magic power. He thought that Helmholtz mixed theory with practice and added that he never encountered a less philosophic conception.[2]

Frege's argument is of interest because it raises the question whether an appeal to a psychic automatism is a species of magic. However, it must be kept in mind that if this were the case everything would be magic,because without such an automatism no progress can be made. The objection, that in such a view theory was mixed with its applications, is without foundation, because propositions concerning the properties of expressions are true or false in the same degree as are propositions concerning the properties of numbers. Actually thoughts about the numbers denoted by the signs belong to the realm of interpretation. However, Frege was right in so far as at the time when he wrote it was not possible to reduce the propositions of arithmetic to propositions concerning expressions.

Frege pointed out that E. Heine and J. Thomae formulated very clearly the formalistic conception of arithmetic recently developed by Hilbert. The difference between the views of Heine and Thomae lay in the fact that Heine regarded numbers as signs, while Thomae rejected as meaningless the problem : what are numbers ?

Frege characterized Thomae's position as follows :

> " In arithmetic only the signs of numbers are necessary ; but these signs are not treated as such, but as forms. The rules in accordance with which these forms are employed are also necessary. We do not learn these rules from the meanings of the signs but posit them on our own authority, reserving the right and acknowledging no need to justify them, although at the time when we exercise this freedom, we keep an eye on possible applications because without applications arithmetic would be a game and nothing more." [3]

[1] Hermann von Helmholtz : " Zählen und Messen erkenntnistheoretisch betrachtet," *Philosophische Aufsätze, Eduard Zeller zu seinem 50-jähr. Doktorjub. gewidmet,* cited by Gottlob Frege : *Grundgesetze der Arithmetik,* Bd. ii, 1903, Jena, pp. 139–140.

[2] Frege : *l.c* , Bd. ii, p. 140.

[3] Frege : *l.c.* p. 102.

This position was undoubtedly motivated by the obvious observation that only propositions concerning signs, at least those formulated in writing, can be completely determined.

If an appeal is made to the counting of pebbles or nuts, or the measurement of segments, it is no longer true that operations are being confined to that which can be formulated. In the first case the reader must have had certain experiences which can be pointed to only in a general way. In the second case wholly fictional objects are being employed.

However, it is true that arithmetic must be constructed with a view to its subsequent application to the measurement of segments.

Because signs can be counted the arithmetic of natural numbers can be constructed without appealing to pebbles or apples. However, the situation is different in the case of measurement. A science about real numbers can be constructed which requires only two fundamental concepts, the concept of a sign and the concept of a class.[1] However, in making this construction the properties of segments must be analysed in terms of these concepts. This means that segments are not discussed but rather relations between signs (natural numbers) are constructed in such a way that they imitate certain properties of segments, namely the properties involved in making geometrical measurements.

The study of real numbers should therefore be preceded by the study of geometrical measurement. This study leads to the semantical conception of number and to familiarity with the fact that no mysterious existences are employed in mathematics. The weakness of this theory lies in the fact that segments are mysterious existences. However, the role of segments is provisional and when the fundamental concepts of arithmetic have been established, the segments will be eliminated. Later such a segment-theory of real numbers will be introduced.[2]

5. The belief in the primacy of natural members which was inherited from the Pythagoreans continued to be very influential during the nineteenth century and has persisted to some extent to the present day. Max Black writes:

> " Weierstrass had tried to demonstrate that all mathematical entities could be developed as constructions of natural numbers ;

[1] Richard Dedekind employs the terms symbol and system. Cf. " The Nature and Meaning of Numbers," *Essays on the Theory of Numbers*, translated by Wooster Woodruff Beman, Chicago, 1901, pp. 44–5.

[2] Cf. 3 6.

Kronecker went further and declared that only the natural numbers were 'real' and that *all* mathematical results must actually be results about the natural numbers. Thus not only were irrational numbers, fractions, and complex numbers never to occur in mathematics, but even negative numbers were taboo. As Kronecker himself said in a striking sentence, which will perhaps bear repetition once more, 'God made the natural numbers; all the rest is man's handiwork'." [1]

The famous French mathematician Henri Poincaré was an ardent advocate of the theory of the primacy of natural numbers but did not ascribe independent existence to them. Poincaré thought that the concept of natural number cannot be reduced to still simpler concepts because no proposition can be constructed without tacitly employing the concepts one and two. He derided those logicians who attempted to construct the concept of the number 1. The following passage will give the reader an idea of Poincare's polemics on this subject:

"I hasten to add, that the definition M. Couturat gives of the number 1 is more satisfactory.

"One, says he, in substance, is the number of elements in a class in which any two elements are identical.

"It is more satisfactory, I have said, in this sense that to define 1, he does not use the word *one*; in compensation, he uses the word *two*. But I fear, if asked what is *two*, M. Couturat would have to use the word *one*." [2]

These arguments are based upon a misunderstanding. It does not follow from the fact that the number 2 can easily be interpreted intuitively, that it is a different kind of object than for example 10^{10}. It is clear that such objects as 10^{10} must be defined because they correspond to nothing in immediate experience. The concept of the number 2 and the concept of the number 10^{10} must therefore be constructed in the same way. Even if it were necessary to employ an intuitive concept of two in such a construction there would be no vicious circle. Actually it is unnecessary to employ such a concept because the symbolical method places adequate means at our disposal.

Despite the marked tendencies and often even uncomfortable narrowness of his ideas, Poincaré's discussion with the

[1] Max Black. *The Nature of Mathematics*, New York, 1934, p. 175.

[2] Henry Poincaré: *Foundations of Science*, translated by George B. Halsted, New York, 1929, p. 458. (Italics inserted by the author.)

logisticians yielded many valuable results. Among the most interesting is his explicit emphasis upon the extensiveness of applications of the principle of *complete or mathematical induction.*

The traditional form of the principle can be formulated as follows:

If 1 has some property and if this property is transitive, every whole number has this property.

In this connection by a *transitive property* is meant one which the number $n+1$ has, if the number n has it.

For example the property of *being greater than a certain number* is such a property.

The understanding of this principle will be facilitated by its application to the following interesting formula:

The symbol $\sum_{i=1}^{n} i^2$ denotes the sum of the squares of the natural numbers from 1 to n inclusive, i.e. it denotes

$$1^2 + 2^2 + 3^2 + \ldots + n^2.$$

The proof of this formula requires that it be verified first for $n = 1$. On the one hand

$$\sum_{i=1}^{1} i^2 = 1.$$

On the other hand: $\dfrac{2+3+1}{6} = 1$. It is now clear that the formula is valid for $n = 1$.

Let it now be supposed that the formula is valid for some natural number p, where the choice of p is entirely arbitrary. In other words let it be supposed that

$$\sum_{i=1}^{p} i^2 = \frac{2p^3 + 3p^2 + p}{6}$$

Under this condition it can be shown that the formula is valid for $p + 1$. In the first place in conformity with the definition of the symbol $\sum_{i=1}^{p} i^2$,

$$\sum_{i=1}^{p+1} i^2 = \sum_{i=1}^{p} i^2 + (p+1)^2.$$

The equality

$$\sum_{i=1}^{p+1} i^2 = \frac{2p^3 + 3p^2 + p + (p+1)^2}{6}$$

$$= \frac{2(p+1)^3 + 3(p+1)^2 + (p+1)}{6}$$

follows from the supposition which has been made.

On the other hand if $(p + 1)$ is substituted for n in the expression

$$\frac{2n^3 + 3n^2 + n,}{6}$$

$$\frac{2(p+1)^3 + 3(p+1)^2 + (p+1)}{6}$$

is obtained.

The given formula is therefore transitive with respect to n. On the basis of the principle of induction it can be inferred, that the formula is true for any natural number.

The nature of this reasoning will not be discussed. The reasoning is altogether convincing here but it is not so convincing when the principle is applied to theorems which cannot be stated as formula.

Let it be supposed that the proof of the following theorem is desired :

If the side of a triangle is divided into $(n + 1)$ equal parts and from each point of division a line is drawn parallel to the second side, the third side will also be divided into $(n + 1)$ equal parts.

Here not a formula but a proposition which is expressed in words is to be proved. It is to this proposition, which will be denoted by the symbol ϕ (n), where n is the letter occurring in the proposition, that the principle of induction will be applied.

The first step in proving this proposition is to verify it for the simplest case, i.e. where the third side is divided into two equal parts. It is clear that in this case n must be taken equal to the number 1. The theorem :

$$\phi\ (1)$$

will be obtained from brief considerations on the congruence of the auxiliary triangles.

To demonstrate the transitivity of the condition ϕ (n) let it be supposed that a certain natural number p, chosen arbitrarily, satisfies the condition, i.e. that ϕ (p) has been established.

The theorem ϕ $(p + 1)$ follows from the supposition and a geometrical construction similar to the preceding one.

It therefore follows that the theorem:

$$\phi\ (n)$$

is valid where n is any natural number.

I have not the slightest doubt concerning this proof, but because it does not conform to the principles of sound reason I would not undertake to explain it to a layman in such a way that he would really understand it. The proof concerns a propositional function, i.e. an expression which contains the letter n and which becomes a proposition if a number is substituted for this letter. But the discussion of literal expressions requires the use of every-day language with all its ambiguities. I do not doubt that on the basis of the illustrations of formulae, pupils willingly believe the teacher when he says that no paradoxes are possible concerning mathematical propositions. But this same pupil will not be too pleased to learn suddenly that he must study grammar as well as mathematics, since at present neither mathematical formulae, nor geometrical constructions are sufficient. If errors are to be avoided the material of investigation must be extended to include certain propositions which as a matter of fact are formulated correctly from the point of view of grammar. But to avoid misunderstandings I explicitly remark that I am not referring to the mere use of propositions. Even in elementary arithmetic propositions are necessary. But the means and the material of investigation differ there. Here a proposition is not only the means but at the same time the object of investigation and this is surely an invasion of the domain of grammar.

The subconscious desire to avoid this unpleasant consequence leads to the tendency to substitute for propositions such symbols as ϕ (n) which was given above. But it is clear that nothing is gained thereby. On the contrary the reader is likely to have new doubts concerning the introduction of a symbol which is unknown to him and the notion of a propositional function which is even less familiar to him. Special training seems necessary but it is well known that suitable training can make us agree to even malicious nonsense.

The situation of classical mathematics is indeed disagreeable. Mathematicians can disdainfully smile at their critics and retain their conviction that no errors result from this state of affairs. But this is small comfort. Mathematicians deride the physicists because they employ simple calculations, but the physicists laugh at the mathematicians because their simple calculations lead them to discoveries which transform the surface of the earth, while the precise demonstrations of the mathematicians are not of much use for this purpose.

Later such primitive induction will be employed in the hope that in the discussion of certain material the reader will agree to accept it as an auxiliary means employed for purposes of orientation. The moment the analysis of the complete system of logic and mathematics is undertaken, it will be seen that all these difficulties can be completely removed. In fact with the introduction of the symbolic language of rational semantics, the principle of complete induction can be formulated much more generally and in such a way that it does not differ intrinsically from the other rules of demonstration.

Poincaré interpreted the principle of induction falsely. He wrote:

> "... This rule, inaccessible to analytic demonstration and to experience, is the veritable type of the synthetic *a priori* judgment. On the other hand, we cannot think of seeing in it a convention, as in some of the postulates of geometry.
>
> "Why then does this judgment force itself upon us with an irresistible evidence. It is because it is only the affirmation of the power of the mind which knows itself capable of conceiving the indefinite repetition of the same act when once this act is possible ..." [1]

Poincaré did not take into account the fact that strictly speaking no logical rule is an analytic proposition because each rule introduces something essentially new. He also neglected the fact that no rule can be proved and that no rule can be regarded as a convention.

The reason for his misunderstanding lay in the fact that he regarded the natural numbers as the only true objects of mathematical analysis.

The famous German mathematician, Professor Hilbert, also succumbed to the lure of whole numbers.[2] In his system whole numbers have an exceptional role, although it is acknowledged

[1] Poincaré: *l.c.*, p 39.

[2] Cf. David Hilbert: "Über die Grundlagen der Logik und der Arithmetik," *Verhandlungen des III Internationalen Mathematiker—Kongresses*, Heidelberg, 1904, pp 174–185, reprinted in *Grundlagen der Geometrie*, 7. Aufl. Berlin, 1930, pp. 247–261.

that they are expressions. But for Hilbert numbers were also the basis of the intuitive method with whose aid he sought to prove the consistency of his system. This proof is conceived in such a way as to rely upon those finitistic reasonings which are necessary to verify a formula which contains the separate whole numbers. Actually it depends upon the application of complete induction, as Poincaré rightly observed.[1]

6. A schoolboy can understand the four operations on whole numbers. He also can understand the reduction of fractions to a common denominator. But the moment he comes to the multiplication of fractions he is blocked and very often can make no progress. Clever schoolboys become convinced that if the teacher says that fractions are to be multiplied by multiplying the numerator by the numerator and the denominator by the denominator, this is the custom and it is necessary to submit to this fate. But schoolboys who are not acute feel instinctively that the teacher is being inconsistent, because up to now he has explained all the operations by using pebbles and apples but he is not explaining multiplication in this way.

Poincaré advised lectures on the theory of proportions first, and thought it desirable to appeal to geometric images.[2] But it is clear that all this is not of much use. Even if this were done no one would ever understand why multiplication of the numerator by the numerator and the denominator by the denominator should be the method of multiplying fractions.

Professor Zaremba defined operations on fractions by appealing to the measurement of segments and thus removed this difficulty.[3] It will be shown that if a sufficient number of illustrations is given, an arithmetic which is based upon the measurement of segments can be constructed in such a way that it becomes a collection of trivial rules which are intelligible to every one. I regard the segment-method as the only sensible method which can be employed which does not require the semantical calculus. Even when this calculus is employed the segment-method retains its value as an auxiliary system for purposes of orientation. All other attempts to develop arithmetic are either fragmentary and therefore not entirely clear, or are based upon certain metaphysical

[1] Poincaré: *l.c*, pp. 465–7.
[2] Poincaré: *l.c.*, pp. 441–2.
[3] Cf. S. Zaremba: *Wstęp do analizy*, Część II (*Introduction to Analysis*, part II), Warszawa, 1918 (hectographed), p. 2.

suppositions which contradict the principles of sound reason. The first objection applies to the axiomatic method which was employed by Peano and Hilbert, the second to such systems as the system of Whitehead and Russell and even to my theory of constructive types,[1] in which it is necessary to presuppose the existence of infinitely many objects without giving an example of even one of them.

Professor Zaremba accepts the theory of natural numbers as something already completed. This supposition is unnecessary because the moment an intuitive idea of segments and their subdivision into parts is accepted, it is possible to construct the theory of natural numbers. It is necessary only to decide to regard natural numbers as expressions and to agree that there are no such individuals as the number 1, the number 2, etc.

The expressions : 1, 2, 3, 4, . . . which belong to the decimal system of natural numbers will be called *natural numbers*.

The class of expressions of the decimal system is precisely determined in the popular sense of this word. There is some reason to believe that in every case in which such expressions are employed, it can be shown that they really belong to the decimal system. There is no need to be concerned with expressions which are written indistinctly or which are too long. Consequently they will not be considered.

Let it be supposed that the usual segments which are known from geometry can be employed and that the method of comparing them and of constructing new segments by combining them, are known.

If it is supposed that the method of constructing a segment n times as long as a given segment E, is known, the resultant segment will be denoted by the symbol $(n\,E)$. In particular the segment $(1\,E)$ is simply the segment E. If m and n are two whole numbers such that in the decimal system m appears immediately before n, the segment $(n\,E)$ is obtained by adding the segment E to the segment $(m\,E)$.

The comparison of natural numbers reduces to the comparison of relative segments. For example :

$$m > n$$

means that the segment $(m\,E)$ is longer than the segment $(n\,E)$ for any E.

The addition of natural numbers reduces to the construction of relative segments.

[1] *T. C. T.*, *T. C. T.*, ii.

The expression $(m + n)$ is a number equal to the natural number which is obtained by measuring with the help of segment E the segment constructed by subtracting the segment $(m\,E)$ from the segment $(n\,E)$.

Multiplication is obtained if it is observed that the segment $(\alpha\,E)$ can in its turn be employed as a unit of measure. This observation permits such segments as $(\beta\,(\alpha\,E))$ to be obtained. For example the segment $(3\,(5\,E))$ is obtained as follows: first a segment five times as long as E is constructed and then a segment three times as long as the new segment. It is clear that the latter segment is equal to the segment $(15\,E)$.

The product $\alpha.\beta$ can therefore be regarded as the number obtained by measuring the segment $(\beta\,(\alpha\,E))$ by means of the segment E.

The same method permits the calculation of differences and quotients with little effort.

Dr. Nikodym does not regard sums, differences, products, and quotients as numbers, but as certain expressions whose *values* are numbers.[1] In practice this distinction makes no difference but in theory it becomes necessary to distinguish between numbers and the results of operations as well. The latter distinction might lead to the illusion that numbers have ideal existence.

The theory of rational numbers is a natural extension of this theory of natural numbers. If α and β are natural numbers $\frac{\alpha}{\beta}$ is said to be a *rational* number.

The following additional rule is accepted:

If E is a segment and α and β are natural numbers, then $\left(\frac{\alpha}{\beta}E\right)$ is a segment obtained as follows:

First the segment I is marked off in such a way that the segment $(\beta\,1)$ is equal to the segment E. Then the segment $(\alpha\,I)$ is constructed. The expression $\left(\frac{\alpha}{\beta}E\right)$ denotes the segment $(\alpha\,I)$. The definitions which have been given for natural numbers apply without change to rational numbers except that the quotients of any two rational numbers can be constructed.

[1] Cf. Otto Nikodym: *Dydaktyka matematyki czystej* (*The Teaching of Pure Mathematics*), Warszawa, 1930, p. 84.

The rules governing operations are derived from those which have previously been given and no other rules are required.

Without discussing the general theory the building up of $\frac{1}{2} \cdot \frac{5}{3}$ will now be considered.

The problem reduces to the construction of the segment $\left(\frac{1}{2}\left(\frac{5}{3}\right)\right)$ with the help of the rules which have been given.

First it must be confirmed that this segment is equal to the segment $\left(\frac{5}{10}\left(\frac{10}{6}\frac{E}{-}\right)\right)$. It is necessary to divide the segment E into six equal parts and to take ten such parts. The division of the segment so obtained $\left(\frac{10}{6}\frac{E}{-}\right)$ into ten equal parts yields the segment $\left(\frac{1}{6}\frac{E}{-}\right)$. If five of these parts are taken, the result is $\left(\frac{5}{6}\frac{E}{-}\right)$. In this way the equality

$$\frac{1}{2} \cdot \frac{5}{3} = \frac{5}{6}$$

is obtained.

Thus the method of Professor Zaremba solves the annoying problem of the multiplication of fractions in a very simple fashion.

The concept of irrational numbers will be explained with the help of the following example:

The problem is to measure the hypotenuse $B\,A$ of the isosceles triangle $A\,O\,B$ by means of the leg $O\,A$. If the relative measure is to be denoted by x,

$$A\,B = (x\,O\,A).$$

If a right isosceles triangle is constructed in such a way that $A\,B$ is one of the legs and if the hypotenuse is denoted by $A\,C$,

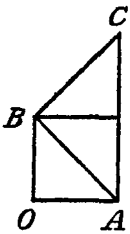

FIG. 2.

$$A\,C = (x\,A\,B)$$

Consequently $A\,C = (x\,(x\,O\,A))$

On the other hand

$$A\,C = 2\,O\,A$$

If then the rule of multiplication which has been accepted is to be applied to this example, it must be agreed that

$$x^2 = 2.$$

A new sign $\sqrt{2}$, which will be regarded as a number, will therefore be accepted and the relation between this number and the rational numbers is fixed by the equation : $(\sqrt{2})^2 = 2$. The construction given makes it certain that this number differs from all the rationals.

Operations on irrational numbers are obtained by applying automatically the rules established for rational numbers.

Clearly there is no important difference between rational and irrational numbers. Each is a certain kind of sign which is employed in measuring segments. The only difference between them lies in the fact that there is no general construction rule for irrational numbers.

A construction with ruler and compass cannot be made in all cases. Large aggregates of segments, for example the segment equal to the circumference of a circle, cannot be marked off in this way. But there is no reason to allow only constructions made with ruler and compass. If, for example, a piece of string is superimposed on the circumference of a circle and the string unfolded, from the physical point of view this construction is equally as good as a construction made with ruler and compass. In each case different rules are required and a system of rules covering all cases cannot be given.

Dedekind attempted to dispose of the matter by accepting the *axiom of continuity*.[1]

Let the following construction rule for the segments $A_n B_n$ be posited :

1. To any natural number, 1, 2, 3, 4, . . . corresponds a segment

$$A_1 B_1, A_2 B_2, A_3 B_3, A_4 B_4, \ldots$$

2. Each subsequent segment is contained within the preceding one.

3. Among the segments $A_n B_n$ may be found one which is smaller than any of the previous segments.

In conformity with the axiom of continuity the segments will have one and only one common point M which will be called the common *limit* of the points A_n and B_n.

[1] Cf. Richard Dedekind: "Continuity and Irrational Numbers," *Essays on the Theory of Numbers, l.c.*, p. 11.

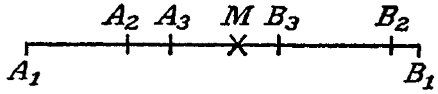

FIG. 3.

If numbers are co-ordinated with the points A_n and B_n in such a way that to the point A_n there will always correspond a number less than the number corresponding to the point B_n, and if to the point M a certain real number is assigned, the latter number is the *upper limit* of the numbers co-ordinated with the points A_n and at the same time is the *lower limit* of the numbers co-ordinated with the points B_n.

Generally speaking the upper limit of the numbers satisfying a certain condition is the smallest number which is greater than the numbers determined by this condition.

If among the given numbers there is a greatest, it is said to be their upper limit.

Analogous definitions can be given for the lower limit.

For example $\sqrt{2}$ is the upper limit of the rational numbers whose squares are less than 2 and at the same time is the lower limit of the rational numbers whose squares are greater than 2. The number π is the upper limit of the semi-circumferences of the polygons inscribed in a circle of radius 1, and is also the lower limit of the semi-circumferences of the polygons circumscribed about this circle.

The method of Dedekind is seen to be very general. However, a certain rule which governs the construction of the numbers must be presented. The sole question to be discussed concerns the methods of constructing such rules. Only the science of semantics can lead to the answer to this question.

The concept of the upper limit is sufficient to understand the foundations of the infinitesimal calculus. However, fundamental questions concerning the *extraction* of roots and other methods of approximating the real numbers must be discussed. These methods lead students to feel that $\sqrt{2}$ and π are unknowns of a certain kind whose approximation is sought, but which cannot be accurately determined. This state of affairs results in a complete confusion of concepts and the illusion that the attainment of infinity is desired. Actually $\sqrt{2}$ is equally as good a sign as 2 and its approximation depends upon the setting up of certain inequalities.

7. The number 0 can be defined as a point. But an intuitive basis for the definition of operations on 0 cannot be obtained

in this way. These operations must have a conventional character because it is impossible to talk about measuring a point by means of a segment or about measuring a segment by means of a point. It is accepted as a convention that:

$$(x + 0) = x \text{ and}$$
$$(x \,.\, 0) = 0.$$

Division by 0 is rejected and fractions are defined in such a way that they cannot have 0 as their denominator, i.e. the symbols $(x : 0)$ and $\frac{x}{0}$ are held to be meaningless.

There is a group of philosophers who on the basis of ontological categories, would like to do away with this restriction. This can be done without any difficulty. In the system of Whitehead and Russell for example $\frac{x}{0}$ is the null class. Nevertheless this restriction cannot be eliminated if only the elementary concepts of geometry and arithmetic are accepted. Without a prior theory of classes attempts of this kind are of no advantage and are unnecessary. Furthermore they contribute to the creation of the illusion that mathematics can be constructed without regard to the concept of an expression and in particular without regard to the concept of *meaningless* expressions. It should immediately be noted that all attempts to construct a system of mathematics without regard to these concepts cannot succeed. There is to-day no system of expressions in which every expression has a determinate meaning and it is vain to hope to have such a system. Moreover why should such a system be sought? Such an attempt may gratify the desires of metaphysicians of the type of Plato, but runs counter to the fact that the concepts to be considered are not themselves clearly defined. Consequently they have no clear meaning and can only obtain such through man's intervention.

The transformation of segment-arithmetic into algebra requires only the substitution of the concept of a vector parallel to an arbitrarily chosen straight line for the concept of a segment.

If this substitution is made 1 is denoted by a certain vector j, -1 by the vector which has the same length but the opposite direction. The relation between two vectors which have the same direction is expressed by a positive number; the relation between two vectors with opposite directions is expressed by a negative number.

The addition and subtraction of numbers reduces to the corresponding operations on vectors; multiplication is performed by applying the rule given for segments to vectors. This rule enables one to prove with ease that $(-3. -5) = 15$. Consequently this result which was always regarded by students as artificial and strained is presented as a natural phenomenon.

If any vector j is selected, a vector a equal to the vector $(-3\,j)$ can be constructed by applying the given rules. It is three times as long as j but has the opposite direction. If the vector a is taken to be the unit, the vector b which is equal to the vector $(-5\,a)$ can be constructed. It is five times as long as a but has the opposite direction. It follows from what has been said that the vector b is fifteen times as long as the vector j and has the same direction. Thus the vector b has been shown to be equal to the vector $(15\,j)$.

It follows from the definition which has been given that $(-3\ .\ -5) = 15$.

8. The discovery of the fact that entirely sufficient construction rules cannot be given for real numbers is due to Jules Richard. Richard formulated his conception as follows:

> "Let us consider everything which we can write down if we write only *n* letters. Inasmuch as there are twenty-six letters in the French alphabet, if we write down *n* letters, this is an *n*-uple arrangement of these twenty-six letters, provided that the same letter can be repeated. The number of these arrangements, is, as is proved in combinatory analysis, twenty-six raised to the *n*th power. All these arrangements can be enumerated. It is necessary only to order them alphabetically. Let us take the double arrangements first, then the triple arrangements, then the quadruple . . . and then the *n*-uple arrangements . . ., ordering them in every case; in this ordered sequence, any *n*-uple arrangement will occupy a certain place however great *n* may be. The majority of these arrangements will not even form words. Let us erase them. Let us also erase those which form a meaningless collection of words. The remaining arrangements will form meaningful propositions . . ." [1]

If among these propositions all the propositions which do not define real numbers less than 1, are removed, a sequence

[1] Jules Richard: *Sur la philosophie des mathématiques*, Paris, 1903, pp. 107–8.

of propositions which define the real numbers less than 1 will remain.

When such a sequence is given, a new real number which is not defined by any of the propositions in this sequence, can be defined by employing the so-called diagonal method of Cantor. If the number defined by the nth proposition of the sequence has the figure k in the nth place, where k is different from 9, the new number will have $k + 1$ in the nth decimal place. If the number defined by the nth proposition has the figure 9 in nth place, the new number will have the figure 0 in the nth place.

If it is assumed that this last proposition is found in the sequence, the so-called paradox of Richard immediately results.[1] Richard recognized such an assumption as inadmissible because it involves a vicious circle. Actually if the above construction is to have meaning it must be supposed that the sequence of propositions is determined and consequently this sequence cannot be a member of the sequence itself. It is difficult to remove this contradiction from every-day language. Poincaré's own definition of the concept of *predicative* propositions, i.e. those which do not refer to themselves, was itself criticized as not being predicative.[2] However, this difficulty can be removed if a suitably constructed language is employed.

From this state of affairs Jules Tannery inferred that there must exist real numbers which cannot be defined in a finite number of words.[3] Such a conclusion is clearly metaphysical. It presupposes the ideal existence of numbers only some of which can be known. If such an assumption is accepted the conception of a mathematics based upon sound reason must be abandoned.

Henri Poincaré, the greatest mathematician of that time, came out immediately against such a concept of mathematics with great vehemence.

He very positively asserted that real numbers do not form a determined class and that it is meaningless to speak of all real numbers. In the spring of 1909 I heard him deliver a lecture at Göttingen, in which he expressed his point of view very emphatically although in bad German. The youthful and

[1] Jules Richard: "Les principes des mathématiques et le problème des ensembles," *Revue générale des sciences pures et appliquées*, vol. 16, 1905, pp. 541–3. Cf. Henri Poincaré: *l.c.*, p. 480.

[2] Cf. Poincaré: *l.c.*, p. 481. Ernst Zermelo: "Neuer Beweis fur die Möglichkeit einer Wohlordnung," *Mathematische Annalen*, Bd. 65, 1908, p. 117.

[3] Richard: *Sur la philosophie des mathématiques*, *l.c.*, p. 113.

brilliant Hermann Weyl, an acknowledged disciple of Hilbert, was present at this lecture. Undoubtedly it influenced his conception of real numbers.

Poincaré's words made a strong impression upon me. A few years after that Göttingen experience, I plucked up enough courage to come out openly against those who accepted numbers not definable in a finite number of words. I considered the system of Whitehead and Russell, but recently published at that time, as contradictory only because it permitted a proof of the existence of the universal class of real numbers.[1] This view was unquestionably too radical. The system of Whitehead and Russell contains no contradiction. Similarly there should be no fear that the supposition of the existence of the universal class of real numbers might lead to contradiction. Not until later did I understand that the situation here is comparable to the story of the guardian angel and the devil. No contradiction develops from Russell's assumption because the guardian angel prevents its occurrence. The occurrence of a contradiction would be due to the fact that the devil has gained the upper hand. If this is kept in mind no contradiction will ever occur. In any case contradiction is not concerned here. What is involved is that a system of mathematics which assumes the existence of no non-constructive objects be developed.[2] I have been concerned with this problem over a period of years, as have W. Hetper and J. Herzberg. Later the reader will be able to judge whether we have succeeded in solving it.

As mathematics developed these matters were neglected. The great majority of mathematicians became reconciled to non-constructive numbers. While an imposing series of results was obtained on the basis of this hypothesis, the presence of metaphysical speculations remained a fundamental flaw.

If it is asked whether the development of mathematics in this way was of any value, without hesitation my answer would be in the affirmative. There is always time for the reconstruction of foundations and for criticism. If it were desired to analyse every detail and to depend only upon oneself, no progress would ever be made. However, provisional results should not be overrated, and it should be recognized that in time they must be precisely and systematically formulated, and be subjected to thorough criticism.

[1] Cf.my article, *Z.S.*, pp. 316 ff. [2] Cf. Richard . *l.c.*, p. 111.

There is no general concept of real numbers and there never will be one. It is a waste of time to discuss this fact. But it by no means follows that a pattern for the creation of real numbers cannot be given and that it will always be necessary to employ intuition. It will be seen later that it is possible to construct patterns which can be applied automatically to the construction of higher and higher types of real numbers by means of operations previously defined. In turn the theory of these patterns can be included in a pattern. It will also be seen that this method leads to a system of mathematics which is as rich as the system based upon the supposition that the class of real numbers exists. The transition from these patterns to the universal class of real numbers will always prove to be possible provided that it is assumed that there exist certain expressions which are not constructible in this system.

If this method of formalization is employed, obviously a conception of mathematics contradictory to that of Poincaré results.

Poincaré was impressed by the Kantian idea of synthetic *apriori* judgments and believed that mathematical intuition can never be included in a pattern. Consequently he stubbornly opposed constructions which are based upon the use of logical symbolism. But he did not indicate how to treat the foundations of modern mathematics and he was forced to reject the whole theory of classes of Cantor. He ignored the fruitfulness of this imposing structure merely because of his unsubstantiated fear that contradiction would result.

The Dutch mathematician Brouwer made an attempt to construct the foundations of mathematics entirely on the basis of Poincaré's doctrine.

Confining himself only to time, Brouwer created the conception of neo-intuitionism which does not deal with a space separated from the time of Lobaczewski. In his inaugural address at the University of Amsterdam in 1912, Brouwer held that:

> "This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness."[1]

[1] Cf. Max Black. *l.c.*, p. 189. This book contains an accurate summary of Brouwer's views.

Thus Brouwer begins with metaphysics and can never extricate himself from it.

In particular I am referring here to his conception of real numbers. Real numbers are the members of infinite sequences of the integers 0, 1, . . ., 9. In infinite sequences of integers chosen at random, the sequences can be regarded as constructed by successive arbitrary choices, each of which is completely independent of the previous choices.[1] The method of *extracting* square roots, which is taught at school, is the height of precision in comparison with this conception. Cantor's ideal class of all real numbers seems clear and crystallized in comparison with the chaos which characterizes Brouwer's obscure ideas. Yet, in the name of clearness of concepts, Brouwer undertook a critique of the foundations of the elementary logical calculus. He substituted for the banal rules derived from daily life, a fantastic logic in which indifferent as well as true and false propositions appear. I will have occasion to discuss Brouwer's views later. Here I only wish to remark that despite the defects which have been mentioned they were not wholly fruitless because they greatly contributed to the rise of that new science called metamathematics, which proved to be very fruitful.

Weyl's metaphysical way of thinking is even more marked than that of Brouwer. Yet his conception of real numbers is very similar to the one at which I will arrive later. Weyl maintains that real numbers can be obtained only by creating a hierarchy of constructive rules. He considers absurd the view that the continuum or the class of real numbers is something completed.[2] Weyl's conception seems very important to me because he differs from the mathematicians who do not wish to consider the limitations of their views but prefer to abandon their goal, the consistent construction of the foundations of arithmetic. Yet Weyl's constructive method is not so precise that it can effectively oppose this view of the mathematicians. If some sacrifice is to be made, something must be obtained in exchange. Either free construction of numbers or a formal system whose rules are not ambiguous is necessary. My system which was based upon the theory of constructive types [3] and which permits the reconstruction of all classical

[1] Cf. Black · *l c.*, p 203.
[2] Cf. Hermann Weyl · " Über die neue Grundlagenkrise in der Mathematik," *Mathematische Zeitschrift*, Bd. 10, 1921, pp. 39–79.
[3] *T. C. T.*, *T. C. T*, ii.

mathematics [1] satisfies the latter condition. But this system is not completely satisfactory, because it requires an unlimited number of verbal definitions and does not permit the proof of the theorem that there exist infinitely many natural numbers to be obtained. The system of rational metamathematics, which will be introduced later, removes these difficulties.

[1] Cf. *M. L.*

CHAPTER IV

THE ELEMENTARY CONCEPTS OF SEMANTICS

1. Systematic investigations concerning the structural properties of mathematical expressions were initiated by Hilbert. He wrote:

> " Mathematics like all other sciences cannot be established by means of logic alone: rather, something is already given us in the imagination as a condition preliminary to the use of logical inference and the application of logical operations; prior to all thought there are certain extra-logical concrete objects which are present intuitively as immediate experience. In order that a logical inference may be certain, these objects must be completely surveyed in all their parts and their mode of production, their differences, their succession or their juxtaposition with the objects is at the same time given directly and intuitively as something, which cannot be reduced to anything else, or as something, which requires no such reduction . . . " [1]

Later he adds:

> " And in particular, in mathematics, the concrete signs themselves whose form is immediately clear and recognizable, are, as a consequence of our arrangement the object of our consideration." [2]

With this in mind, Hilbert supplemented the axioms of logic by purely mathematical axioms and regarded both types of axioms as simple juxtapositions of signs. His only concern was to show that the theorem: $1 = 0$ is not one of the theorems which can be obtained from these axioms by applying purely formal rules of procedure.

To show this, the structure of mathematical expressions obviously had to be analysed. In particular such questions as the following had to be examined: what expression will be obtained if the expression G is substituted for the expression F in a given expression E? Hilbert gave no systematic construction rules for expressions; he was satisfied with general

[1] David Hilbert: " Die Grundlagen der Mathematik," *Abhandlungen des mathematischen Seminars zu Hamburg*, Bd. 6, 1928, p. 65, reprinted as " Anhang IX " in *Grundlagen der Geometrie*, Leipzig and Berlin, 7 Aufl., 1930, pp. 289–290.

[2] *l.c.*, p. 290.

hints, which, however, his fellow-workers (Bernays, Ackermann, von Neumann) later made more precise. These intuitive considerations with regard to the construction of expressions, which at every step involve induction either openly or in a concealed form, were only apparently simpler than the problems of mathematics itself. Actually some new science, at least as profound and with as far reaching consequences as mathematics, was required.

I resolved to formalize this science. At first I based it upon two fundamental concepts, the concept of the ordered juxtaposition of two expressions and the concept of substitution, $(E\ F\ G\ H)$, which is to be read:

H is the result of the substitution of G for F in E.[1]

Later the concept of the juxtaposition of expressions proved to be derivable from the pattern, $(E\ F\ G\ H)$.[2] Mr. Hetper later showed that the concept of substitution, i.e. the pattern, $(E\ F\ G\ H)$ can be reduced to an even simpler pattern $\{E\ F\}$, which is to be read:

The expression F is contained in the expression E.[3]

These investigations immediately led to the construction of a system of elementary semantics, i.e. the science of expressions. Later it appeared that mathematics and metamathematics can be constructed with the help of this system. The axioms of elementary semantics are, as will be seen, the usual rules for operating upon expressions, symbolically formulated. Consequently they can be regarded as the most elementary and natural statements which can be constructed.

The intuitive semantics, upon which Hilbert and his school rely in their investigations, raises many questions requiring further consideration. A number of judgments varying in clarity and precision can be made concerning expressions. The questions which arise, because letters are not written with sufficient clarity, have already been pointed out. At every step questions of this kind may be raised. For example it may be asked whether given copies of the letter *a* are alike or whether they differ. Because there are no two identical copies of the letter *a*, serious difficulties may be encountered at the very beginning. To talk about the class of letters *a* rather than about the letter *a*, as some people wish to do, does not remove

[1] *N. G. L. M.*, p. 704.
[2] *N. G. L. M.*, II, p. 527.
[3] Cf. W. Hetper: "Podstawy semantyki" ("The Foundations of Semantics"), *Wiadomości Matematyczne*, t 43, Warszawa, 1936, pp. 57–86.

the difficulty because, while one can deal with separate copies of the letter *a*, the class of letters *a* is a confused and unclear concept, since *a* cannot always be distinguished from *o* or *d*.

It follows from the discussion between Professor Bernays and Mr. Müller,[1] concerning the Hilbert conception of expressions, that the use of an intuitive approach in reflections upon expressions can lead to such complicated results, that instead of producing simplicity and clarity, a hopeless chaos seems to be characteristic of investigations concerning the form and individuality of signs. This discussion teaches that even with regard to the simplest objects one cannot say with impunity anything which enters one's mind. It is necessary to confine oneself to purely practical statements concerning the problem of the construction of expressions from given elements. Only thus can the clarity and simplicity sought be obtained.

During his sojourn in Cracow, Professor Zermelo remarked to me that expressions are much more complicated structures than numbers or classes and that many serious questions may be raised in connection with them. I answered that this would undoubtedly be so, if one wished to enter upon metaphysical investigations. But if one confines oneself to constructing new expressions by applying previously given rules to given expressions, doubt is simply pedantry.

It should be added that the use of signs in no way depends upon one's concept of reality. It makes no difference whether signs are regarded as things, collections of atoms, or expressions. Since visions are one type of signs, the latter might be regarded as ideal existences. A vision and a cult connected with it can be associated with signs as well as with concepts. If one has investigated the individual properties of signs, one can even come to believe in the existence of signs which cannot be written. In any case it is possible to talk about expressions which cannot be constructed with the help of a given system of signs and rules. However, it must be kept in mind that in so doing one oversteps the bounds of sound reason. It is possible to talk significantly only about specified signs or about words which can be constructed from them by applying given rules. The concept of a sign as such is very confused and vague. Everything can be regarded as a sign, but no thing

[1] Cf. Paul Bernays · "Erwiderung au' die Note von Herrn Aloys Müller: 'Über Zahlen als Zeichen,'" *Mathematische Annalen*, Bd 90, Berlin, 1923, pp. 159–163.

by itself is a sign. Everything depends upon the convention adopted in advance.[1]

2. Elementary semantics is not concerned with any expressions whatsoever. The concept of any expression and the concept of any name whatsoever is completely undetermined and cannot be the object of precise study.

To be able to talk of expressions it is necessary first to give the rules of their construction. Expressions which constitute the subject-matter of semantics will be called *proper expressions*. Proper expressions can be classified as either constant or variable expressions.

First the construction rules for constant expressions will be given:

I. **0** is a constant expression.

II. If E and F are constant expressions, then $* E F$ is a constant expression.

With the help of these rules as many constant expressions as desired can be constructed.

It can easily be confirmed that the only expression which can be constructed containing no stars is **0**.

Of course there is no way of proving that § for example is not a constant expression, but it is easily seen that the sign § is not one of the expressions which can be constructed. It differs from **0**, contains no stars, and consequently cannot be obtained by applying either rule I or rule II.

It may easily be confirmed that it is possible to construct only one constant expression with one star, namely:

$$*\mathbf{0}\mathbf{0},$$

only two constant expressions with two stars, namely:

$$*\mathbf{0}*\mathbf{0}\mathbf{0},\quad **\mathbf{0}\mathbf{0}\mathbf{0},$$

only five constant expressions with three stars, namely:

$$***\mathbf{0}\mathbf{0}\mathbf{0}\mathbf{0},\quad *\mathbf{0}**\mathbf{0}\mathbf{0}\mathbf{0},\quad **\mathbf{0}*\mathbf{0}\mathbf{0}\mathbf{0},$$
$$*\mathbf{0}*\mathbf{0}*\mathbf{0}\mathbf{0},\quad **\mathbf{0}\mathbf{0}*\mathbf{0}\mathbf{0}.$$

and only twelve constant expressions with four stars, namely:

$$\begin{array}{ll} ****\mathbf{00000}, & *\mathbf{0}***\mathbf{0000}, \\ **\mathbf{0}**\mathbf{0000}, & *\mathbf{0}*\mathbf{0}**\mathbf{000}, \\ ***\mathbf{0}*\mathbf{0000}, & *\mathbf{0}**\mathbf{0}*\mathbf{000}, \\ **\mathbf{0}*\mathbf{0}*\mathbf{000}, & *\mathbf{0}*\mathbf{0}*\mathbf{0}*\mathbf{00}, \\ **\mathbf{0}*\mathbf{00}*\mathbf{00}, & **\mathbf{00}*\mathbf{0}*\mathbf{00}, \\ ***\mathbf{000}*\mathbf{00}, & **\mathbf{00}**\mathbf{000}, \end{array}$$

[1] Cf. the metaphysics of signs in the article of Jan F. Drewnowski: "Zarys program filozoficznego" ("Outline of a Philosophical Program"), *Przegląd filozoficzny*, rocznik, 37, 1934, pp 3–38.

It should be observed that this method has nothing in common with the traditional method of definition. Neither the *genus* nor *differentia specifica* of the concept of a constant expression can be given. Only the fact that one can construct constant expressions is essential. The situation is more or less like that of any tradesman. For example, while no printer can say what a book is nor prove that an advertisement is not a book, any printer can print as many books as material conditions permit.

The above rules will now be supplemented in such a way that variable as well as constant expressions can be constructed.

The following rules are posited:

III. If E is a constant expression, E is a proper expression.
IV. The letters u, v, w, . . . are *semantical letters.*
V. If E is a semantical letter, E is a proper expression.
VI. If E and F are proper expressions, $* E F$ is a proper expression.
VII. If E is a proper expression which contains the semantical letter I, E is a variable expression.

These rules permit the construction of the following variable expressions:

$$* u \mathbf{0}, \quad * * u v w, \quad * \mathbf{0} * u \mathbf{0}, \text{ etc.}$$

Thus the material dealt with in elementary semantics has been determined.

It can easily be shown that this symbolism is uniquely determined.[1]

In proving this theorem, the following formulation of the principle of complete induction is employed:

If $\mathbf{0}$ *and the letters have some property, and if when the proper expressions E and F have this property, the proper expression E F also has this property, then all the proper expressions which can be constructed by means of the stipulated rules have this property.*

The proof is based upon the theorems of the arithmetic of natural numbers and the topological properties of proper expressions. For example, the fact that if E contains m stars and F contains n stars, $* E F$ contains $m + n + 1$ stars is accepted as one of the premises.

Pedantically precise proofs will not be insisted upon here. Precision is possible only in semantics itself. Here the degree

[1] Mr. Skarzeński proved this theorem in 1928.

of precision which can be obtained in ordinary arithmetic must suffice.

To prove the uniqueness of the system of expressions which has been given, it will first be shown that every expression which has been constructed in accordance with the stipulated rules and which contains n stars, contains $n + 1$ letters.

If E is a letter or zero, E is seen to contain 0 stars and $0 + 1$ letters.

It will now be supposed that this law is valid for the expressions E and F.

If it is now supposed that the expression $* E F$ has been constructed, on the basis of the suppositions which have been made, E, for example, will have K stars and $k + 1$ letters, while F will have m stars and $m + 1$ letters. $* E F$ is seen to have $m + k + 1$ stars and $(m + k + 1) + 1$ letters. Thus it is clear that this property is transitive.

If the principle of complete induction, as formulated above, is invoked the theorem will have been proved for the general case.

Clearly this proof like all intuitive proofs is of interest only for orientation.

It will now be proved that every segment of an expression from the right side contains more letters than stars. An inductive proof will be given.

If E is a letter or zero, every segment of the expression E contains less stars than letters.

It will now be supposed that the theorem is true for the expressions E and F.

Let any segment of the expression $* E F$ from the right side be considered. If this segment does not contain part of the expression E, it is a segment of the expression F and therefore contains less stars than letters. If it does contain part of the expression E, this part is a segment of the expression E and therefore contains less stars than letters. Since F also contains less stars than letters, the whole segment contains less stars than letters.

On the basis of this theorem it can easily be proved that if E, F, E', and F' are expressions and if $* E F \equiv * E' F'$, then $E \equiv E'$ and $F \equiv F'$. The sign $\equiv$ denotes semantical identity here.

For if E' for example were longer than E there would be a certain segment R such that $E R = E'$. However, since R must contain less stars than letters, R would have at most

n stars and $n + 1$ letters. On the other hand because E' has p stars and $p + 1$ letters, E must have $p - n$ stars and $p - n$ letters, which is precluded by the previous theorem.

3. The construction of the rules of substitution of constant expressions for constant expressions in constant expressions will now be considered.

The proposition: **(** $E\ F\ G\ H$ **)** *is true*, is to be read as follows: *The constant expression H is the result of the substitution of the constant expression G for the constant expression F in the constant expression E.*

The uniqueness of this pattern follows immediately from the considerations of the last article. For example in the proposition: **(** ∗**0**∗**00**∗**000**∗**00** **)** the values of the expressions E, F, G, and H respectively can only be the expressions ∗**0**∗**00**, ∗**00**, **0** and ∗**00** respectively. No other reading is possible.

The way in which these operations are to be performed will be established by means of recursive rules as was the method of construction of constant expressions. First the method of substitution in the expression **0** will be established. Then substitution in the constant expression ∗ $E\ F$ will be reduced to substitution in the constant expressions E and F.

(*a*) The following rules of substitution in the expression **0** are to be accepted:

I. If F is **0**, then **(** **0** $F\ G\ G$ **)** *is true.*
II. If F is not **0**, then **(** **0** $F\ G$ **0** **)** *is true.*

Theoretically these rules do not decide the question of the uniqueness of substitution; however, in practice they always yield a unique result, for there are no other methods of making a substitution in the expression **0**.

These rules obviously include the case when G is substituted for a proper expression different from **0**, in the expression **0**. Clearly the substitution must be fictional in this case and cannot produce any change in the expression **0**. Consequently **0**, which was given initially, is regarded as the result of the substitution.

(*b*) If **(** $E\ F\ G\ H$ **)** and **(** $E'\ F\ G\ H'$ **)** *are true*, the following two cases occur:

I′. If F is the expression ∗ $E\ E'$, then **(** ∗ $E\ E'\ F\ G\ G$ **)** *is true.*
II′. If F differs from ∗ $E\ E'$, then **(** ∗ $E\ E'\ F\ G$ ∗ $H\ H'$**)** *is true.*

Because only expressions constructed by successive application of the operator ∗ $E\ F$ to expressions previously constructed, can be dealt with, it is clear that substitution is performed in every concrete case and in but one way.

The next question to be answered is: what is the result of the substitution of the expression 0 for * 0 0 in the expression * 0 * 0 0? The answer will be formulated with the help of the pattern (* 0 * 0 0 * 0 0 0 H) and is obtained by successively making substitutions in the process of constructing the expressions * 0 * 0 0.

First, on the basis of rule II, it is seen that

(0 * 0 0 0 0) *is true.*

If 0 is taken for E, G, H, E' and H' and * 0 0 for F, on the basis of rule I′ it follows that (* 0 0 * 0 0 0 0) *is true.*

If 0 is taken for E, G, H and H' and * 0 0 for E' and F, on the basis of rule II′ it follows that * 0 * 0 0 differs from * 0 0. It may then be concluded that

(* 0 * 0 0 * 0 0 0 * 0 0) *is true.*

It may easily be ascertained that no other result can be obtained by examining the structure of the expression * 0 * 0 0.

The theorems:

(0 0 0 0),
(0 * 0 0 * * 0 0 0 0),
(0 0 * 0 0 * 0 0),
(* 0 0 * 0 0 0 0),
(* 0 0 0 * 0 0 * * 0 0 * 0 0),
(* 0 * 0 * 0 0 * 0 * 0 0 * 0 0 * 0 * 0 0)

may be investigated in an analogous way.

4. Expressions of the form (E F G H) where E, F, G and H are any proper expressions will now be considered. If at least one of the proper expressions E, F, G or H contains a semantical letter the expression (E F G H) will be called a *propositional function* and the semantical letters contained in it will be called *real variables*. It is clear that functions of one, two, three, etc., variables can be constructed in accordance with the number of semantical letters to be found in the proper expressions E, F, G, and H. A theory of substitution for semantical letters in a propositional function is unnecessary, because such substitutions will be made only in concrete cases in which a semantical letter really occurs in the propositional function. If, for example, the expression E is substituted for the variable u in the propositional function (* 0 u * 0 0 0 u) in such a way that the expression E is merely

written for the variable u, the expression $(\ast \mathbf{0} E \ast \mathbf{0}\mathbf{0}\mathbf{0} E)$ is obviously obtained. The expression obtained by substituting a proper expression for a real variable in a propositional function is called the value of the *propositional function.* Obviously the values of a propositional function can be either propositional functions or *propositions.* The latter case occurs when and only when the value of the propositional function does not contain a semantical letter. The rules which have been given enable one to determine whether the propositions which are values of semantical functions are true or false. If true they will be characterized by the letter $\mathbf{V}$, if false by the letter $\mathbf{\Lambda}$.

The analogy between algebraic equations and inequations and these propositional functions is evident. Constant expressions which, when substituted in a propositional function make this function a true proposition, will be called the *roots* of the propositional function. As a result of the conception of dead substitutions, it may be said that all constant expressions are roots of the proposition $(E\,F\,G\,H)$, which has no semantical letters, if this proposition is true. No constant expression is a root of this proposition if this proposition is false. It can be ascertained that propositional functions can have either a finite or infinite number of roots, or no roots, and that all constant expressions can be roots of propositional functions.

Propositional functions will be investigated with the help of tables.

(A) If the function $(u\,\mathbf{0}\,\mathbf{0}\,u)$ is considered it is seen to become a true proposition if $\mathbf{0}$ is substituted for u. If it is supposed that $(E\,\mathbf{0}\,\mathbf{0}\,E)$ and $(G\,\mathbf{0}\,\mathbf{0}\,G)$ *are true*, $\ast E\,G$ cannot be identical with $\mathbf{0}$ for in every case the result contains at least one star. Then on the basis of rule II′ it follows that $(\ast E\,G\,\mathbf{0}\,\mathbf{0} \ast E\,G)$ *is true.* On the basis of the principle of induction it follows that all constant expressions are roots of the function $(u\,\mathbf{0}\,\mathbf{0}\,u)$.

This state of affairs may be represented by the following table:

u	$(u\,\mathbf{0}\,\mathbf{0}\,u)$
E	$\mathbf{V}$

(B) It may easily be seen that no constant expression is a root of the function $(\mathbf{0}\,\mathbf{0}\,\mathbf{0} \ast u\,\mathbf{0})$. This follows simply from the observation that no substitution of a constant expression for u in $* u 0$ can yield 0 as its result because in every case at least one star will be contained in the result. This state of affairs would contradict rule I.

To illustrate this case, the following table is employed:

u	$(000 * u 0)$
E	Λ

(C) The determination of the roots of the function $(u * 000 * 00)$ will now be attempted.

Clearly 0 is not a root of this function (cf. rule II). Consequently if this function has a root, the root must be some expression $* E F$. Then either E and F are identical with 0 and therefore by rule I′, $* E F$ is not a root of the given function, or E and F are not both identical with 0 and therefore by rule II′ $(E * 0000)$, and $(F * 0000)$ *are true*. These conditions can be satisfied only when E and F are 0 or $* 00$. In the first case rule II is applied; in the second, rule I′ is applied. Since rules I and II′ cannot be applied here there are three possibilities: either E is 0 and F is $* 00$, E is $* 00$, and F is 0 or E and F are both $* 00$. The case where E and F are both 0 has already been rejected. This state of affairs may be illustrated by the following table

u	$(u * 000 * 00)$
$* 0 * 00$	V
$* * 000$	V
$* * 00 * 00$	V
. . . .	Λ

(D) It will now be shown that all constant expressions except $* 00$ and $* * 000$ are roots of the function $(* * 000 u 0 * * 000)$. It may first be seen that $* * 000$ is not a root of this function because if rule I′ is applied a different result is obtained. If E is not $* * 000$ the application of rule II′ yields the expressions:

$(* 00 E 0 * 00)$ and $(0 E 00)$ which *are true*.

Two possibilities now present themselves. If E is $* 00$, then in view of considerations analogous to those of a moment ago, E is not a root of the function $(* 00 u 0 * 00)$. If E is not $* 00$, this function reduces to the function $(0 u 00)$ which is satisfied by any substitution for u, since when 0 is substituted for u, rule I may be applied, and when another expression is substituted for 0, rule II may be applied. The values of this function are therefore seen to be true.

The following table illustrates this case:

u	$(**000u0**000)$
$*00$	Λ
$**000$	V
$\ldots\ldots\ldots$	Λ

The function $(**000u0**000)$ thus leads to the determination of all the expressions *contained* in the expression $**000$ with the exception of $\mathbf{0}$. Actually in addition to $\mathbf{0}$, the expression $**000$ contains only the expressions $*00$ and $**000$, i.e. the expressions which are not roots of the given function.

Similarly the function $(*0*0*0*00u0*0*0*0*00)$ is satisfied by all expressions with the exception of $*0*0*0*00$, $*0*0*00$, $*0*00$ and $*00$, which are all the expressions contained in the expression $*0*0*0*00$ with the exception of $\mathbf{0}$.

The proposition: $\{E\,F\}$ *is true*, will signify that the expression F is contained in the expression E. Later it will be seen that the symbol $\{E\,F\}$ may be constructed with the help of the symbol $(E\,F\,G\,H)$ and the symbols of the elementary calculus of propositions.

(E) Examples of functions whose roots form a sequence will now be considered.

If the function

$$(*uu*000u)$$

is considered, $\mathbf{0}$ is seen to be a root of this function. (Cf. rule I′.)

If $*\,E\,F$ is a root of this function when rule II′ is applied, the conditions: $(*EF*000E)$ and $(*EF*000F)$ *are true*, are obtained.

E must be identical with F if the operation of substitution is to be uniquely determined. Consequently $(*EE*000E)$ must *be true*.

If $*\,E\,E$ is a root of the function, then in conformity with rule II′, $**EE*EE$ is also a root of the function. The function is therefore seen to be *transitive* with respect to the substitution of $*\,E\,E$ for E. On the basis of the principle of complete induction it follows that the roots of this function form the sequence:

$$0,\ *00,\ **00*00,\ ***00*00**00*00,\ \ldots$$

It can be ascertained analogously that the roots of the functions

$(*0u*000u)$ and

$(*u0*000u)$

form the sequences

$\mathbf{0}$, $\ast\,\mathbf{0\,0}$, $\ast\,\mathbf{0}\ast\mathbf{0\,0}$, $\ast\,\mathbf{0}\ast\mathbf{0}\ast\mathbf{0\,0}$ . . . and

$\mathbf{0}$, $\ast\,\mathbf{0\,0}$, $\ast\ast\,\mathbf{0\,0\,0}$, $\ast\ast\ast\,\mathbf{0\,0\,0\,0}$, . . . respectively.

(F) A function of two or more semantical letters corresponds to equations in two or more unknowns.

(a) If the function $(\,u\,\mathbf{0\,0}\,v\,)$ is considered, the following table may be constructed:

u	v	$(\,u\,\mathbf{0\,0}\,v\,)$
E	E	$\mathbf{V}$
$\mathbf{0}$	$\ast\,E\,F$	$\mathbf{\Lambda}$

This table is not complete because the method of constructing the pattern of an expression which differs from a given expression E is unknown. Nevertheless the roots of the function are seen to be those and only those pairs of constant expressions whose elements are identical. Consequently $(\,E\,\mathbf{0\,0}\,F\,)$ *is true* may be read: *E is identical with F.* Later $(\,E\,\mathbf{0\,0}\,F\,)$ will be abbreviated by the symbol $=E\,F$.

(b) The function $(\,E\,v\,\mathbf{0}\,E\,)$ may easily be seen to characterize all the expressions contained in any given expression E with the exception of $\mathbf{0}$.

$(\,\mathbf{0}\,v\,\mathbf{0\,0}\,)$ is seen to be satisfied by all constant expressions. No expressions other than $\mathbf{0}$ is said to be contained in $\mathbf{0}$.

If it is supposed that the functions $(\,E\,v\,\mathbf{0}\,E\,)$ and $(\,F\,v\,\mathbf{0}\,F\,)$ are satisfied by all expressions with the exception of those contained in E and F which are different from $\mathbf{0}$ and if the expression $(\,E\,F\,v\,\mathbf{0}\ast E\,F\,)$ is constructed, two cases may be distinguished. If v is $\ast\,E\,F$, only rule I′ which requires the identity of $\mathbf{0}$ and $\ast\,E\,F$ may be applied. Since this identity cannot occur, it is obvious that $\ast\,E\,F$ is not a root of the given function.

If v is not $\ast\,E\,F$ the matter reduces to the conditions: $(\,E\,v\,\mathbf{0}\,E\,)$ and $(\,F\,v\,\mathbf{0}\,F\,)$ *are true*, which determine all the expressions contained in $\ast\,E\,F$ with the exception of $\mathbf{0}$ and $\ast\,E\,F$. Therefore the given function leads to the determination of all the expressions contained in $\ast\,E\,F$ with the exception of $\mathbf{0}$.

5. The arithmetic of whole numbers will now be considered. The system of the arithmetic of whole numbers constructed by Hetper,[1] will be the basis of this discussion.

The expression $(\,\ast\,E\,E\ast\mathbf{0\,0\,0}\,E\,)$ will be abbreviated by

[1] W. Hetper: "Semantische Arithmetik," *Sprawozdania z posiedzeń towarzystwa naukowego*, Wydział III, t. 27, 1934, pp. 9–26.

the symbol **Integ** E and will be read: *E is a whole number.* The numerical expressions of the decimal system will be regarded as abbreviations of certain constant expressions in accordance with the following table:

Abbreviation	Expression
1	* 0 0
2	* 1 1
3	* 2 2
4	* 3 3
5	* 4 4
6	* 5 5
7	* 6 6
8	* 7 7
9	* 8 8
$\overline{10}$	* 9 9
$\overline{11}$	* $\overline{10}$ $\overline{10}$

The reader should observe that a vinculum is placed above two-figured numbers and above all other multi-figured numbers in order to avoid ambiguity.

The comparison of whole numbers reduces to the problem of inclusion.

The proposition: *E is greater than or equal to F* is said to be the abbreviation of the proposition:

E and F are whole numbers and $\{ E\ F \}$, *are true.*

This proposition is denoted by the symbol $\geqslant$ E F *is true.* I shall employ a method of notation similar to that of Łukasiewicz,[1] although it may displease the reader who is accustomed to place the sign $\geqslant$ between E and F. I see no reason for abandoning the notation of Łukasiewicz merely because the ordinary notation is more familiar. Symbolical language requires the elimination of so many habits, that there is no reason to give the habit of using the ordinary method of notation special consideration.

The proposition: *E is greater than F* is the abbreviation of the proposition: $\geqslant$ E F *is true,* and it is not the case that $=$ E F *is true.*

This proposition is denoted by the symbol $>$ E F *is true.*

Theorem I:

If **Integ** E *is true,* if $\{ E\ F \}$ *is true,* either $=$ E F or $\{ E * F\ F \}$ *is true.*

[1] Jan Łukasiewicz: *Elementy logiki matematycznej* (*The Elements of Mathematical Logic*), lithographed, Warszawa, 1929, p. 40.

Proof: The second part of the antecedent and the consequent of this theorem will be abbreviated by the symbol ΦE. By applying the principle of complete induction Φ **(0)** *and if* Φ (E), *then* $\Phi(* E E)$ will be proved. This simplified formulation of the property of transitivity is proved by the fact, that if **Integ** $* E K$, then $* E K$, which was confirmed above. The letters employed stand for whole numbers and therefore are to be regarded as constant expressions, which require no special discussion here.

(*a*) If $\{ \mathbf{0} F \}$ *is true*, then $= F \mathbf{0}$ *is true* and consequently Φ **(0)**.

(*b*) Suppose, that $\Phi(E)$, i.e. if $\{ E F \}$ *is true*, then either

$= E F$ or $\{ E * F F \}$ *is true.*

If now $\{ * E E F \}$ *is true*, then either $= * E E F$, or $\{ E F \}$ *is true.*

If $= * E E F$, *is true*, then $\Phi(* E E)$ because one of the two postulated conditions is satisfied.

If $\{ E F \}$ *is true*, it follows from the supposition that either $= E F$ or $\{ E * F F \}$ *is true.*

If $= E F$ *is true*, then $\{ * E E * F F \}$ *is true*, whence it follows that $\Phi(* E E)$ since the second condition is satisfied.

If $\{ E * F F \}$ *is true*, it follows that $\{ * E E * F F \}$ *is true* and therefore $\Phi(* E E)$ because the second condition is satisfied.

Thus the transitivity of the condition $\Phi(E)$ has been confirmed.

A reader who is not accustomed to precise reasonings will think that this proof is very complicated. However, it must be kept in mind, that the difficulty of the proof lies in the fact that a series of conditions must be distinguished. Because there are many conditions, it is easy to make a mistake and the impression that the proof is difficult arises. Actually very trivial observations are involved.

Theorem II:

If **Integ** E *is true*, then if **Integ** F *is true*, either $\{ E F \}$ or $\{ F E \}$ *is true.*

Proof: Here again the second part of the antecedent and the consequent will be denoted by $\Phi(E)$ and once again it will be proved that Φ **(0)** and: *if* $\Phi(E)$, *then* $\Phi(* E E)$.

(*a*) If F is any constant expression, one has $\{ F \mathbf{0} \}$ *is true*, therefore the second alternative is always satisfied. Consequently Φ **(0)** for any F.

(*b*) Suppose that $\Phi(E)$, i.e. if **Integ** F *is true*,

either $\{ E F \}$ or $\{ F E \}$ *is true.*

If $\{ E F \}$ *is true*, then $\{ * E E F \}$ *is true*, and therefore $\Phi(* E E)$.

If $\{ F E \}$ *is true*, then it follows from **Integ** F *is true* and from theorem I that either $= F E$ *is true* or $\{ F * E E \}$ *is true*.

If $= F E$ *is true*, then $\{ * E E F \}$ *is true* and consequently once again $\Phi(* E E)$.

If $\{ F * E E \}$ *is true*, the second alternative is satisfied and again in this case $\Phi(* E E)$.

The condition $\Phi(E)$ has been shown to be transitive.

It follows from conditions (*a*) and (*b*) on the basis of the principle of complete induction that the theorem is true.

The addition of whole numbers is denoted by the symbol $+ E F G$. The proposition : $+ E F G$ *is true* is to be read : *G is the sum of the whole numbers E and F*. This proposition is the abbreviation of the proposition :

Integ E *is true*, **Integ** F *is true and* $(* E E \mathbf{1} F G)$ *is true*.

The reduction of multiplication of whole numbers to addition will now be exhibited.

The proposition : $\times E F G$ *is true*, which is to be read : *G is the product of the whole numbers E and F*, is introduced.

For example $\times \mathbf{3}\, \mathbf{4}\, \overline{\mathbf{12}}$ is written instead of the traditional $\mathbf{3} \,.\, \mathbf{4} = \overline{\mathbf{12}}$.

To construct the semantical proposition which corresponds to the symbol $\times \mathbf{3}\, \mathbf{4}\, \overline{\mathbf{12}}$, the finite sequence of pairs of whole numbers :

$* \mathbf{3}\, \mathbf{1}$, $* \mathbf{6}\, \mathbf{2}$, $* \mathbf{9}\, \mathbf{3}$, $* \overline{\mathbf{12}}\, \mathbf{4}$ is formed.

The first pair is seen to contain in its first place the multiplicand and in its second place the number $\mathbf{1}$, and the last expression has in its first place the product and in its second place the multiplier. Every expression $* C D$ different from $* \mathbf{3}\, \mathbf{1}$, which belongs to the sequence is constructed from a certain expression $* A B$ which belongs to the finite sequence such that $+ A \mathbf{3} C$ *is true* and $+ B \mathbf{1} D$ *is true*.

It is clear that $* \overline{\mathbf{12}}\, \mathbf{4}$ may be found in this sequence only if the latter contains $* \mathbf{9}\, \mathbf{3}$, and $* \mathbf{9}\, \mathbf{3}$ may be found in this sequence only if the sequence contains $* \mathbf{6}\, \mathbf{2}$, the pair which is derived from $* \mathbf{3}\, \mathbf{1}$ by performing the indicated operations.

It is easy to see that this finite sequence can be obtained by constructing a certain constant expression.

If the expression :

$$* * * * \mathbf{3}\, \mathbf{1} * \mathbf{6}\, \mathbf{2} * \mathbf{9}\, \mathbf{3} * \overline{\mathbf{12}}\, \mathbf{4}$$

which is denoted by $\mathbf{W_0}$ is constructed, in it the pairs which have been mentioned, can all be found, but such pairs of numbers as $* \overline{\mathbf{11}}\,\overline{\mathbf{11}}$, $* \overline{\mathbf{10}}\,\overline{\mathbf{10}}$ may also be found in this expression because every whole number with the exception of $\mathbf{0}$ has been defined as a pair of whole numbers.

Hetper[1] introduced still another expression in order to avoid this ambiguity.

In the case being discussed, from $\mathbf{W_0}$ and the pairs which have been mentioned, the following expression may be constructed :

$$* * * * \mathbf{W}_0 * \mathbf{3}\,\mathbf{1} * \mathbf{W}_0 * \mathbf{6}\,\mathbf{2} * \mathbf{W}_0 * \mathbf{9}\,\mathbf{3} * \mathbf{W}_0 * \overline{\mathbf{12}}\,\mathbf{4}.$$

This expression will be denoted by $\mathbf{L}_0$.

If now, one speaks of such pairs of expressions $* A\ B$ as enter into $\mathbf{L}_0$ by means of the expression $* \mathbf{W}_0 * A\ B$, all ambiguity is removed. Actually pairs which are contained in whole numbers are eliminated because they are not included by this pattern. For the same reason such pairs as for example :

$$* * \mathbf{W}_0 * \mathbf{3}\,\mathbf{1} * \mathbf{W}_0 * \mathbf{6}\,\mathbf{2}$$

are eliminated.

A definition of multiplication will now be given. Where E and F are natural numbers $\times\ E\ F\ G$ is the abbreviation of the following proposition : If either E or F are $\mathbf{0}$, G equals $\mathbf{0}$; if E and F differ from $\mathbf{0}$, a certain constant expression L has the following properties :

1. There is a constant expression W such that

$$\{ L * W * E\,\mathbf{1} \}\ \textit{is true.}$$

2. There is an expression M such that

$$= L * M * W * G\ F\ \textit{is true.}$$

3. If $\{ L * W * C\ D \}$ *is true*, where C and D are constant expressions such that it is not the case that $= * C\ D * E\,\mathbf{1}$ *is true*, there are certain constant expressions A and B that $\{ L * W * A\ B \}$ *is true* and $+\ A\ E\ C$ and $+\ B\,\mathbf{1}\,D$ *are true.*

This definition seems rather intricate because the expression L cannot be given explicitly but must be described. In a symbolical system this description is obviously a certain semantical expression. Here I have confined myself to a literal representation.

6. The introduction of rational numbers presents no

[1] W. Hetper : " Podstawy semantyki," *l.c.*, p. 74.

difficulties. They are simply pairs of expressions constructed as follows:

Abbreviation	Expression
$\frac{1}{1}$	$* 1 * 0 1$
$\frac{1}{2}$	$* 1 * 0 2$
$\frac{2}{1}$	$* 2 * 0 1$

In these pairs the numerator is found in the first place while a pair of numbers composed of **0** and the denominator occupies the second place. Clearly the latter pair can never be a whole number because the denominator is never **0**. The whole fraction can never be a whole number because the first member of the couple is always a whole number and the second member is never a whole number.

The equality of rationals cannot be reduced to semantical identity.

The proposition: $\underset{\mathbf{R}}{=} E\,F$ *is true,* which is to be read: *The rational numbers E and F are equal,* is introduced. This proposition is regarded as the abbreviation of the following proposition: *There are whole numbers A, B, C, D and G such that it is not the case that* $= \mathbf{0}\, B$ *is true and* $= E * A * \mathbf{0}\, B$, $= F * C * O\, D$, *and* $\times\, A\, D\, G$ *and* $\times\, B\, C\, G$ *are true.*

Obviously use is made here of the ordinary criss-cross multiplication.

Inequality may be defined analogously.

The addition of rational numbers may be reduced to the addition and multiplication of whole numbers as may the multiplication of rational numbers.

Subtraction and division of rational numbers may be reduced to the subtraction and multiplication of whole numbers; however, it is stipulated that division by **0** is precluded.

7. While it is true that this method does not permit the construction of sums, differences, products and quotients directly, it permits the construction of the symbols: $\underset{\mathbf{R}}{+} E\,F\,G$, $\underset{\mathbf{R}}{-} E\,F\,G$, $\underset{\mathbf{R}}{\times} E\,F\,G$, $\underset{\mathbf{R}}{:} E\,F\,G$. Consequently the moment the logical calculus is constructed the arithmetic of rational numbers may be regarded as a chapter of elementary

semantics, provided that the principle of complete induction, conceived as a construction rule for theorems, is added to it.

The following objection to the construction which has been given in this chapter must be answered. It is maintained that in order to prove the uniqueness of the structure of expressions, the principle of complete induction and the concept of a whole number must be employed, and consequently the construction of a whole number involves a vicious circle. This objection is only apparently correct. The proof of the uniqueness of this construction of expressions is based upon an intuitive concept of a whole number but is merely an auxiliary proof and not an integral part of the system. The moment this construction has been shown to be unique the rules of the system can be formulated, and these rules in turn permit the construction of the formal arithmetic of whole numbers.

Whether or not this formal arithmetic is covered by intuitive arithmetic cannot be decided since it is not precisely defined. Nevertheless it may be observed that it makes no difference whether one writes:

$$\mathbf{0},\ *\mathbf{0}\,\mathbf{0},\ **\mathbf{0}\,\mathbf{0}*\mathbf{0}\,\mathbf{0},\ ***\mathbf{0}\,\mathbf{0}*\mathbf{0}\,\mathbf{0}**\mathbf{0}\,\mathbf{0}*\mathbf{0}\,\mathbf{0}\ \text{etc., or}$$
$$\mathbf{0},\ 1\ ,\ 1+1\ \ 1+1+1\ \text{etc.,}$$

and that the rules governing the operations upon them in every case give results which are compatible with the results of ordinary arithmetic.

Chapter V

THE CALCULUS OF PROPOSITIONS

1. The calculus of propositions originated during the Middle Ages. Credit should be given Professor Łukasiewicz [1] for having discovered pertinent data mentioned in Prantl's great history of logic which no one previously had appraised.

In the first place it is worthwhile to observe the following example obtained from the works of Albert the Great (1193–1280):

I. "It does not follow: *Every rose is intelligible, therefore all which is a rose is intelligible,* for if one posits that there are no roses, the antecedent would be true but the consequent false." [2]

Albert affirms that one can make no valid inference, if the antecedent is true but the consequent false. The problem of null classes, which as was seen Aristotle did not consider, is also of interest here. If it is kept in mind that Aristotle neglected this problem, even though our contemporary Sleszyński [3] tried to justify him, the importance of Albert's idea must be acknowledged.

The following rules of reasoning may also be found.

II. "If *B* follows from the conjunction of *A* and some necessary proposition, *B* follows from *A* alone." [4]

III. "Implications of the following type are formally admissible: *Socrates exists and Socrates does not exist, therefore a walking-cane stands in the corner.*" [5]

IV. "The denial of the antecedent follows from the denial of the consequent." [6]

All these rules were subsequently formulated symbolically and included as theorems of the calculus of propositions.

[1] Cf. Jan Łukasiewicz: "Zur Geschichte der Aussagenlogik," *Erkenntnis*, vol. 5, 1935–6, pp. 111–131.

[2] Cf. Carl Prantl: *Geschichte der logik im Abendlande*, Leipzig, 1870, Bd. 4, p. 74, n. 285, no. 16.

[3] Jan Sleszyński: *Teoria dowodu* (*Theory of Proof*), tom 1, Kraków, 1925, p. 123 f.

[4] Prantl: *l.c.*, p 73, n 285, no. 7.

[5] Prantl: *l.c.*, pp. 73–4, n. 285, no. 8.

[6] Prantl: *l.c.*, p. 73, n. 285, no. 4.

Rule II introduces the logical product: *A* and *B*.

Rule III affirms that a false proposition implies any proposition. This rule is especially important. At first sight it seems a little strange. It will be seen later that it is entirely natural.

Rule IV gives the so-called principle of conversion which is of great significance in the calculus of propositions.

In the *Dyalectice introductiones Illuminati doctoris et martyris Raymundi Lulli* may be found the principle of the elementary syllogism. Lull (1234–1305) was the widely known creator of the logical machine.

I. "In every valid formal implication, that which follows from the consequent follows from the antecedent." [1]

The following important rules may also be observed:

II. "Every implication whose antecedent contradicts the denial of the consequent is valid." [2]

III. The so-called principle of logical identity: "Implication from one synonym to another . . . is valid." [3]

Because of its self-evident character this most banal of all principles was perhaps most difficult to understand.

Duns Scotus (1265 ?–1308) formulated the following rules:

I. "From any impossible proposition any other proposition follows, not on the basis of formal implication but on the basis of valid, simple [5] material implication." [4]

II. "Every true proposition follows from any other proposition on the basis of valid material implication, *ut nunc*. . . ." [5, 6]

The fundamental idea of these rules was already to be found in Albert the Great. But the novel element in them is the introduction of the concept of *material* implication as distinct from *formal* implication. In practical life one employs only formal implication, which relies on the passage from one proposition to another connected with it in certain ways on the basis of its form. However, if the truth values of a set of

[1] Prantl: *l c.*, 1867, Bd. 3, p. 141, n. 623.
[2] *l.c.*, pp. 141–2, n. 623.
[3] *l.c.*, p. 142, n. 623.
[4] *l c.*, p. 141, n. 621.
[5] The distinction between a simple implication and an implication *ut nunc* is not usually drawn in modern logic. The former is an implication which holds universally irrespective of the time at which the statement is made. The latter is one which holds only at the time when the statement is made. (Translators' note.)
[6] *l c.*, p. 141, n. 622.

propositions are given, namely, if with regard to each of them one can decide whether it is true or false, one can also speak of material implication. This matter will be treated later in detail.

The examples here cited give irrefutable evidence that the Scholastics went far beyond Aristotle and created the foundations of what to-day is called the calculus of propositions. They were unable to construct a calculus because they lacked both the conception of a perfect system of logic and the symbolic language. Consequently their ideas did not influence the further development of logic.

Nevertheless the *Ars Magna* of Raymund Lull may be regarded as the first outline of a system of logic, although still confused. In that work the problem is to arrange three concepts as three concentric circles and to form from them all possible combinations. This is analogous to the construction of an astrological horoscope and thus to rank magic; none the less there is here a recognition of the need to mechanize logical operations.[1]

Contemporary historical investigations have shown that the development of logic advanced systematically, gradually conquering constantly new domains. And as Vaillati noted the sixteenth century Jesuit, Clavius, discovered the following principle of reasoning in the *Elements* of Euclid:

If from the denial of a proposition the truth of this proposition follows, then this proposition is true.

Łukasiewicz discovered that in the writings of the Polish Jesuits, in particular in Adam Krasnodębski's *Philosophia Aristotelis explicata, Varsavia*, 1676, this principle was highly praised and called *Consequentia mirabilis*.[2]

A clear idea of the system of symbolic logic was not developed until the seventeenth century.

It originated in the ideas of Joachim Jungius who noted that the implication:

If A is the father of B, then B is the son of A,

is not reducible to syllogistic form. Thus Jungius definitively overthrew Aristotle's authority and exerted a fruitful influence

[1] Cf. Otto Hölder: *Die mathematische Methode*, Berlin, 1924, p. 4.

[2] Cf. Jan Łukasiewicz: "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls," *Comptes rendus des Séances de la Société des Sciences et des Lettres de Varsovie*, Classe III, vol. 23, 1930, p. 67.

on his contemporaries.[1] Dalagranus (*Ars Signorum*) and Wilkins (*Essay towards the Real Character*) [2] were the first to attempt to apply symbolic language to logic. But Leibniz as was noted above was the first to work out a clear conception of the system of symbolic logic. The works of Leibniz deal with the calculus of classes; they do not, however, go beyond fragmentary attempts.

The disciples of Leibniz, Lambert, and Segner [3] continued these efforts but their work had no direct influence on the development of the calculus of propositions.

The real creator of the calculus of propositions was the English mathematician, *George Boole* (1815–1864).

Concerning Boole's discovery, Sleszyński writes as follows:

> "Boole's thought arose on the basis of the symbolism peculiar to English mathematicians of the first half of the last century. This symbolism is employed especially in algebra, where all the calculi can have different interpretations. According to the practice of these mathematicians we operate with certain symbols without regard to their meaning. All the properties of algebraic symbols then result from their definitions and vary with the different kinds of algebras. But these different properties of symbols become evident only indirectly; for one can prove certain fundamental theorems common to all kinds of algebras, from which likewise follow the rules of the calculus, also common to all kinds of algebras. Boole as a mathematician frequently adopted this symbolical point of view in his excellent and very original works. We will shortly see what mark this point of view left on logic." [4]

And later [5]:

> "In discussing the subject matter of logic Boole notes, that logic deals with two kinds of relations: the relations between things and the relations between facts. Here Boole is introducing the distinction between the logic of classes and the logic of propositions. Thus Boole calls propositions of the first kind primary, and propositions of the second kind secondary. . . . In the logic of propositions he introduces the concept which Frege later dealt with as a logical function."

In addition to Boole one should mention De Morgan, whose

[1] Cf. Federigo Enriques: *The Historic Development of Logic*, translated by Jerome Rosenthal, New York, 1929, p 118.
[2] Enriques: *l c.*, p. 82.
[3] Enriques: *l.c.*, p. 164, pp. 98–100.
[4] Sleszyński: *l c.*, Tom. II, 1929, p. 13.
[5] Sleszyński: *l.c.*, pp. 14–15.

works had great influence on the development of the calculus of propositions.

MacColl made the first attempts to construct the logic of propositions (*The Calculus of Equivalent Statements*).[1] During the same period W. S. Jevons, C. S. Peirce, J. Venn, E. Schröder[2] worked on the foundations of logic. The joint efforts of these investigators led to the rise of what is to-day called the *algebra of logic*.

The algebra of logic was not intimately related to life and science. It was entirely neglected by official science. The opinion of Kant,[3] that Aristotle's logic was a completed and closed work, had a restraining influence on nineteenth century philosophers. They did not believe in the value of logical investigations and restricted their efforts to psychological analysis of the process of exact thinking and to epistemological problems (J. S. Mill, C. Sigwart, W. Wundt, and others).[4]

Reflecting on this development it is difficult not to imagine that some mysterious force manifests itself here, which is intrinsic to human minds, and leads to attempts to destroy efforts at thorough-going criticism. It is universally admitted that before modern science could attain its present position, it had to exert great effort in combatting medieval prejudice. Medievalism defended itself in terms of the principles of scholastic logic. Perhaps just because of that logic has become a synonym of retrogression and unproductiveness and has been abandoned. One must note, however, that the identification of logic with scholastic logic led to great confusion of concepts. Vague psychology of thinking was identified with logic and the creative efforts of the logicians were completely disregarded and logic was considered a mental plaything.

There is no doubt that perhaps just because nineteenth century logic was cut off from the main line of scientific development, it was a purely formal construction. The logical calculus received great elaboration at Schröder's hands, but none the less it is not related in any way with either every day thinking or with scientific thinking. The problem of

[1] Cf. Alessandro Padoa. "La logique déductive dans sa dernière phase de développement," *Revue de Métaphysique et de Morale*, Paris, année 19, 1911, p. 873.

[2] Enriques: *l c.*, p. 156.

[3] Cf. Sleszyński: *l c.*, Tom I, p. 28.

[4] Cf., e g., Christoph von Sigwart: *Logic*, translated by Helen Dendy, 2nd ed. revised and enlarged, London, 1895, vol. i, p. 20. Wilhelm Wundt. *Logik*, 2 Aufl. Stuttgart, 1893, Bd. 1, pp. 1–2.

constructing a *symbolic system of mathematics* based upon the calculus of propositions was first considered toward the end of the nineteenth century. Peano and Frege [1] solved this problem each in his own way, independently of one another. Frege's work was neglected because of his unusually difficult symbolism. Guiseppe Peano, on the other hand, created a very simple and clear symbolism and achieved greater simplicity in the construction of his system. His *Formulaire de mathématiques* [2] became universally known ; it called forth vehement opposition from Poincaré, but great enthusiasm on the part of Couturat and Russell.[3] At the beginning of the twentieth century, a spirited polemic, conducted on a very high plane, resulted. It would seem that discussions of the type carried on by Socrates and Protagoras in Athens have returned. From all this, it is clear that a completely new logic is required and that the vegetation of logic in the antechamber of science has ended once and for all.

But the construction of a system of logic free from contradiction was not accomplished. This problem was not solved even in principle until Whitehead and Russell,[4] although the idea was fascinating to many minds. On the other hand Poincaré, as the representative of traditional intuitionism, was beside himself with anger.

He wrote :

> " Thus, be it understood, to demonstrate a theorem, it is neither necessary nor even advantageous to know what it means. The geometer might be replaced by the *logic piano*, imagined by Stanley Jevons ; or if you choose, a machine might be imagined where the assumptions were put in at one end, while the theorems come out at the other, like the legendary Chicago machine where the pigs go in alive and come out transformed into hams and sausages. No more than these machines need the mathematician know what he does." [5]

In another place Poincaré wrote[6] :

> " The symbolic language created by Peano plays a very grand

[1] Cf. Enriques . *l.c.*, pp. 162–4.
[2] Guiseppe Peano : *Formulaire de mathematiques*, Turin, 1st ed., 1895.
[3] Cf. *Revue de Métaphysique et de Morale*, 1905–1906, 1909–1911, années 13, 14, 17–19.
[4] A. N. Whitehead and B. Russell : *Principia Mathematica*, Cambridge, vol i, 1910
[5] Henri Poincaré . *Foundations of Science*, translated by George B. Halsted, N Y., 1929, p. 451.
[6] Henri Poincaré : *l.c.*, pp. 456–7.

role in these new researches. It is capable of rendering some service, but I think M. Couturat attaches to it an exaggerated importance which must astonish Peano himself.

"The essential element of this language is certainly algebraic signs which represent the different conjunctions: *if, and, or, therefore.*[1] That these signs may be convenient is possible, but that they are destined to revolutionize all philosophy is a different matter. It is difficult to admit that the word *if* acquires, when written $\supset$, a virtue it had when not written *if.*"

Poincaré forgot, that getting rid of ambiguity has very great merit and importance and that a symbolic language does just this. He forgot that there is a great difference between the construction of expressions in terms of certain symbols and the use of everyday language with all its confusions and traps.

"This invention of Peano was first called *pasigraphy*, that is to say, the art of writing a treatise on mathematics without using a single word of ordinary language. This name defined its range very exactly. Later it was raised to a more eminent dignity by conferring on it the title of logistic." [2]

This name angered Poincaré, for it seemed that a revolution in logic had been brought about. Poincaré did not take account of the fact that this name showed excessive modesty. It was as if someone wished to call the old doctrine of Galen *medicine* and contemporary medicine, for example, *medicamentis.*

However, that which to-day is still called *logic* differs from that which is called *logistic* in so far as it is less precisely formulated, does not form a compact whole, and is inseparably connected with the problems of grammar.

The adherents of the old logic make very subtle analyses of the results of present-day logic in order to show that it is not true logic.

Feys, for example, would like to show that the symbolic method cannot reflect accurately what the thought contains.[3] But no one has ever known what the thought contains. Actually everything which the old logic had to say in connection with this problem was pitifully primitive and was full of contradictions. It must be recognized that there is no hope for learning anything about mental processes until the construction of a symbolic logic is begun.

[1] Italics Chwistek's.

[2] Poincaré: *l c.*, p. 457.

[3] Robert Feys: "La transcription logistique du raisonnement. Son interet et ses limites," *Revue Neo-Scholastique de Philosophie*, vol. 27, 1925, pp. 67–8.

Symbolic logic was the inevitable reaction to the banalities and confusions of traditional logic. It is a familiar fact that traditional logic is cultivated even to-day. Whatever the cost may be attempts are still being made to modernize it and to obtain from it something deserving consideration. Time after time these attempts have proved disappointing.

To convince oneself of the truth of these words it suffices to pick up Professor Goblot's logic text, published after the war and thus after the completion of the work of Whitehead and Russell. Professor Goblot completely disregarded this work, since he felt that it belongs to that branch of study which is called *logistic*.

Although he concedes that *logistic* has the merit of having *recognized, derived, and classified the different kinds of constructive operations used in mathematical reasoning*, he maintains that he is concerned with the general laws of thought and not just with mathematical reasoning.[1] He does not take into consideration even for a moment the fact that mathematics must employ general laws of thought and that a system of mathematics cannot be constructed without appealing to these laws. Neither does he observe that Celarent, Camestres, and other similar banalities to which he devotes much space are of no use in the derivation of mathematics and consequently are not general laws of thought. Professor Goblot's book is not a logic—it is a popular journalistic column on the theme of traditional logic. It is a sad commentary on the impermanent and short life of great scientific centres. When Poincaré scoffed at Couturat, he did not realize that he was substituting retrogression toward medievalism for enthusiasm and creative ambition.[2]

However, it must be kept in mind that Poincaré's position was very close to pragmatism and that he took Bergson's irrationalism seriously.[3] He was of the generation of mathematicians who wished to see exact thought confined within the limits of mathematics and he did not realize that if rigorous criticism is not undertaken in connection with all the problems of life, a detrimental confusion of concepts will occur. He did not observe that the old logic has presented and even to-day presents a double danger. On the one hand, crude orientational rules are identified with exact laws of thought

[1] Edmond Goblot · *Traité de logique*, 5ième ed. Paris, 1929, p. xix.
[2] Cf. Poincaré . *l.c.*, p. 450, pp 472–6.
[3] Cf. Poincaré : *l.c.*, *The Value of Science*, Ch 10.

and on the other hand disbelief in exact thought leads to the return of an extremely individualistic irrationalism.

If Aristotle's work is regarded as the beginning of the struggle for a rationalistic view of the world and if the principle of contradiction which he formulated is regarded as a guide-post, by means of which it is continually possible to make advances in the construction of a system of logic, his work is one of the most valuable attainments of the human mind. But if the syllogistic system is taken seriously and if with Kant it is believed that this system is a closed but barren science, serious dangers may result. Under these circumstances it is not surprising that an irrationalist such as Hegel maintains that contradiction is the basis of knowledge, nor that the communistic philosopher Thahlheimer asserts simply that the principle of identity is false because there are no immutable concepts. The pragmatic reaction then must be accepted as completely true because it is difficult to pretend constantly that there are exact foundations of thought if everything is but roughly determined.

If a critique of the old logic is desired, the book of Schiller[1], the pragmatist, is a document of honest sincerity and keen observation. But it is sad that he completely ignores contemporary logic and fails to mention Boole, De Morgan, Peirce, Jevons, Venn, and Whitehead. He mentions Bertrand Russell only incidentally in connection with a certain paradox.[2] Intentionally I have mentioned only famous English logicians, whose names should have been familiar to Schiller if only from book catalogues. It is worth observing that the names of Jevons, Venn, and Couturat are listed on the jacket of Schiller's book, as the authors of a number of works published by MacMillan and Co., Ltd.

Such occurrences seem particularly interesting because they exhibit the danger of a return to barbarism which lies at the basis of present-day culture and which is constantly discussed.

Professor Hölder, who was cited above, was not well acquainted with Whitehead and Russell's *Principia Mathematics*. He dismissed *logistic* with the brief remark that it does not enjoy the recognition of philosophers.[3] Professor

[1] F. C S. Schiller: *Formal Logic*, 2nd ed., London, 1934.
[2] Schiller, *l.c.*, p. 373, n. 2.
[3] Hölder: *l.c.*, p. 272.

Hölder does not hesitate to write that the calculus of propositions not only does not contribute to the solution of logical problems but introduces confusion into them.[1]

This view was propounded over fifteen years ago. To-day such a position is rare. Since the famous mathematician Hilbert became reconciled to the calculus of propositions and in collaboration with Ackermann [2] wrote a very interesting book on the foundations of this calculus, it is difficult to conceive that any mathematician would hazard such a disdainful judgment.

The struggle concerning the conception of the foundations of mathematics still continues but its result can already be foreseen.

Whitehead and Russell's work gave the basis of the calculus of propositions of to-day. Sheffer,[3] Nicod,[4] and Łukasiewicz [5] made important improvements. Sheffer reduced the concepts of the calculus of propositions to one fundamental concept. Nicod reduced the calculus of propositions to one axiom and separated the rules of procedure from the symbolic axioms. Łukasiewicz introduced a symbolic method that makes possible the elimination of the dots and parentheses which are employed to separate expressions. This method proved to be very fruitful, and will be employed in the remainder of the text.

Hilbert and his school employ a symbolism which is not suitable for precise formal calculi. The great merit of this school is that it initiated metalogical investigations, i.e. investigations concerning the properties of a system of symbolic logic.

2. Lectures on the calculus of propositions usually begin with the conventional affirmation that there are true and false propositions. An analysis of the relations between propositions is then undertaken. This method requires that certain allowances be made even at the outset because the propositions which are concerned are not known. If everyday language is employed fundamental difficulties are encountered because, not every proposition of everyday language has a clearly

[1] Hölder: *l.c.*, p 277.

[2] David Hilbert und W. Ackermann: *Grundzuge der theoretischen Logik*, Berlin, 1928, 2 Aufl., 1938.

[3] H. M. Sheffer: "A Set of Five Independent Postulates for Boolean Algebras, with Application to Logical Constants," *Transactions of the American Mathematical Society*, vol. 14, 1913, pp. 481–8.

[4] Jean Nicod: "A Reduction in the Number of the Primitive Propositions of Logic," *Proceedings of the Cambridge Philosophical Society*, vol. 19, 1916–19, pp. 32–41.

[5] Jan Łukasiewicz: *Elementy logiki matematycznej*, *l.c.*

determined meaning and because, since questions might be raised about every proposition considered, it is very difficult to set up the criteria for the truth and falsity of the propositions of everyday language. The possibility still remains that some simply and clearly defined discipline, for example, elementary arithmetic, can be employed. However, it is known that in constructing the foundations of elementary arithmetic in a precise manner, great difficulties are encountered and this construction cannot be accomplished without employing the auxiliary concepts of semantics. Under these conditions pure formalism might seem the only way out of the difficulty. If this were the case, the foundations of logic would go far beyond the bounds of sound reason and the abyss between formal logic and ordinary reasoning would have to be regarded as a sad but inescapable necessity. It will easily be ascertained that this is not the case. It will be seen that it is not hard to give examples of true and false propositions whose meanings are completely and exactly determined. One need only turn to the fundamental concepts of elementary semantics to find such examples.

First the concept of a logical expression will be constructed. The following rules are posited:

1. If E, F, G and H are proper expressions $(E\,F\,G\,H)$ is a logical expression.
2. p, q, r, s, t . . . are logical letters.
3. If E is a logical letter, E is a logical expression.
4. If E and F are logical expressions $/\,E\,F$ is a logical expression.

The new pattern: $/\,E\,F$ which is called a logical pattern will now be introduced. The meaning of this pattern is explained by the following table or matrix:

E	F	$/\,E\,F$
V	**V**	**Λ**
V	**Λ**	**V**
Λ	**V**	**V**
Λ	**Λ**	**V**

This table sets up the following rule: The logical expression $/\,E\,F$ has the value **Λ** when and only when the logical expressions E and F have the value **V**.

In everyday language this expression may be read: not both E and F occur. The pattern $/\,E\,F$ was first proposed by the American logician Sheffer.[1]

[1] Sheffer: *l.c.*, p. 482.

Logical expressions which contain neither semantical nor logical letters will be called elementary propositions. The logical expressions obtained by substituting constant expressions for E, F, G, and H in the pattern $(E\,F\,G\,H)$ are seen to be elementary propositions. Each of these propositions has, as may be seen, either the value $\mathbf{V}$ or the value $\mathbf{\Lambda}$.

If E and F are elementary propositions which have either the value $\mathbf{V}$ or the value $\mathbf{\Lambda}$, $/\,E\,F$ is an elementary proposition which, on the basis of the above table, will have either the value $\mathbf{V}$ or the value $\mathbf{\Lambda}$.

It follows from this observation that every elementary proposition which is constructed must have either the value $\mathbf{V}$ or the value $\mathbf{\Lambda}$, and that the value of a particular elementary proposition can always be determined.

Logical expressions which contain logical but not semantical letters will be called *logical functions*.

Logical expressions which contain both logical and semantical letters will be called *semantico-logical functions*.

Logical expressions which contain semantical but not logical letters will be called *semantical functions*.

Examples of logical functions:

$$/\,p\,p,\ //\,p\,p\,/\,q\,q,\ /\,p\,/\,q\,q,\ //\,p\,q\,/\,p\,q,\ ///\,p\,p\,/\,q\,q\,/\,p\,q,$$
$$/\,(\mathbf{0\,0\,0\,0})\,p,\ /\,p\,/\,(*\,\mathbf{0\,0\,0\,0\,0})\,p.$$

Examples of semantico-logical functions:

$$/\,p\,/\,(u\,\mathbf{0\,0}\,u)\,(u\,\mathbf{0\,0}\,u),\ /\,(u\,\mathbf{0\,0\,0})\,/\,p\,q.$$

Examples of semantical functions:

$$//\,(u\mathbf{000})\,(u\mathbf{000})\,//\,(u\,v\,\mathbf{0}\,u)\,(u\,v\,\mathbf{0}\,u)\,/\,(u\,v\,\mathbf{0}\,u)\,(u\,v\,\mathbf{0}\,u),$$
$$/\,(\mathbf{0\,0\,0\,0})\,(u\,\mathbf{0}\,*\,\mathbf{0\,0\,0}).$$

3. The pattern $/\,E\,F$ proves to be very useful in constructing the fundamental logical concepts.

(*a*) First the negation of a given logical expression will be constructed.

If we consider the truth table:

E	$/\,E\,E$
$\mathbf{V}$	$\mathbf{\Lambda}$
$\mathbf{\Lambda}$	$\mathbf{V}$

it is evident that $/\,E\,E$ has the value $\mathbf{\Lambda}$ when and only when E has the value $\mathbf{V}$. If E has the value $\mathbf{\Lambda}$, $/\,E\,E$ will have the value $\mathbf{V}$. The expression $/\,E\,E$ may therefore be regarded as the negation of the expression E. This expression will be abbreviated by the symbol $\sim E$.

(*b*) The logical sum will now be constructed.

The table of truth values of the pattern $/ / E E / F F$ is as follows:

E	F	$/EE$	$/FF$	$//EE/FF$
V	**V**	**Λ**	**Λ**	**V**
V	**Λ**	**Λ**	**V**	**V**
Λ	**V**	**V**	**V**	**V**
Λ	**Λ**	**V**	**Λ**	**Λ**

Clearly $/ / E E / F F$ has the value **Λ** when and only when the logical expressions E and F have the value **Λ**. Thus the pattern $/ / E E / F F$ corresponds to the algebraic condition: $x \,.\, y = 0$.

Obviously this condition is not satisfied when both x and y differ from 0.

The condition $x \,.\, y = 0$ is read:

$$x = 0 \text{ or } y = 0,$$

where both x and y may equal 0.

This is very much like saying: *Mr. X or Mr. Y will help in connection with this matter* where it is not precluded that both Mr. X and Mr. Y may help.

The pattern $/ / E E / F F$ is abbreviated by the symbol $\mathbf{V}\ E\ F$, which is read: E or F. It is called a *logical sum.*

If now the pattern:

$$/ \sim = F\, \mathbf{0}\, (\, E\, F\, \mathbf{0}\, E\,)$$

is constructed, it follows from the above discussion that this pattern is equivalent to the pattern $\mathbf{V} = F\,\mathbf{0} \sim (\, E\, F\, \mathbf{0}\, E\,)$. As a consequence of this equivalence it follows that the pattern in question has the value **V** when and only when the expression E contains the expression F. Consequently the inclusion pattern $\{\, E\ F\, \}$ which was introduced in the previous chapter may be regarded as the abbreviation of the pattern:

$$/ \sim = F\, \mathbf{0}\, (\, E\, F\, \mathbf{0}\, E\,)$$

A consideration of the pattern $\mathbf{V} = F\,\mathbf{0} \sim (\, E\, F\, \mathbf{0}\, E\,)$ makes it clear that these patterns are entirely compatible.

Actually it is obvious that for any constant expression E, $\{\, E\, \mathbf{0}\, \}$ *is true*, i.e. any constant expression contains **0**. If F is not **0**, the substitution of the expression **0** for F in E gives E if and only if the substitution is a dead one, i.e. if E does not contain F, because a real substitution of the expression **0** for the expression F different from **0**, in the expression **0**, will produce a modification in the expression E.

Thus it is apparent that the substitution-pattern $(\, E\ F\ G\ H\,)$ and the logical pattern $/\ E\ F$ are sufficient for the construction of the concepts of semantical identity and semantical inclusion.

This state of affairs may be illustrated by the following example:

u	~(**000*u*0**000)	= *u*	0 V = *u*0~(**000*u*0**000)
0	Λ	V	V
*00	V	Λ	V
**000	V	Λ	V
....	Λ	Λ	Λ

If an expression different from **0**, ***00** and from ****000** is taken for *u*, it is seen that a dead substitution will be involved and that **(**000*u*0**000)** will become a true proposition and the logical expression **~(**000*u*0**000)** must have the value **Λ**. But because **=** *u* **0** also has the value **Λ**, in this case, **V = 0 ~(**000*u*0**000)** will likewise have the value **Λ**. Thus it has been confirmed that the expression ****000** contains only the expressions ****000**, ***00**, **0**.

(*c*) The pattern **/ /** *E F* **/** *E F*, i.e. **/** *E F* is called the logical product and is abbreviated by the symbol **Λ** *E F*.

The table of truth values of this pattern is as follows:

E	*F*	/ *E F*	/ *E F* / *E F*
V	V	Λ	V
V	Λ	V	Λ
Λ	V	V	Λ
Λ	Λ	V	Λ

This pattern corresponds precisely to what is intended when one says both *E* and *F* occur.

(*d*) The pattern **/ / /** *E E* **/** *F F* **/** *E F*, i.e. **/ V** *E F* **/** *E F* is abbreviated by the symbol **≡**.

This pattern obviously corresponds to what is intended when one talks about the equivalence of two algebraic equations. The table of truth values for this pattern is as follows:

E	*F*	/ *E F*	V *E F*	≡ *E F*
V	V	Λ	V	V
V	Λ	V	V	Λ
Λ	V	V	V	Λ
Λ	Λ	V	Λ	V

Thus **≡** *E F* has the value **V** when and only when *E* and *F* have the same truth value. If *E* and *F* have different truth values, **≡** *E F* has the value **Λ**.

To facilitate the understanding of the meaning of the equivalence pattern it will be applied in solving semantical equations.

The concept of a semantical equation is introduced with the help of the following rules:

(1) If $= E\ F$ is a semantical function, $= E\ F$ is a semantical equation.

(2) If E and F are semantical equations, $/\ E\ F$ is a semantical equation.

A constant expression such that, when it is substituted in a semantical equation this equation will become a logical expression which has the value $\mathbf{V}$, will be called a *root* of this equation.

Two semantical equations which have the same roots will be called *equivalent*.

The following assumptions are sufficient to solve equations in one unknown, i.e. equations which contain only one semantical letter:

$$
\begin{array}{ll}
(1) & \equiv\, = u\,\mathbf{0} = \mathbf{0}\,u \\
(2) & = u\,u \\
(3) & \equiv\, = * u\,v * w\,x \wedge = u\,w = v\,x \\
(4) & \sim\, = \mathbf{0} * u\,v
\end{array}
$$

It is easy to verify that the expressions in these assumptions are semantical equations. The acceptance of these suppositions requires that all constant expressions be roots of equations in one unknown. These assumptions simply describe the operations performed in comparing two proper expressions.

For illustrative purposes the equations:

$$= * * u\,\mathbf{0}\,\mathbf{0} * * \mathbf{0}\,u\,\mathbf{0}$$

will be solved.

Application of assumption (3) yields

$$\equiv\, = * * u\,\mathbf{0}\,\mathbf{0} * * \mathbf{0}\,u\,\mathbf{0} \wedge = * u\,\mathbf{0} * \mathbf{0}\,u = \mathbf{0}\,\mathbf{0} \qquad (a)$$

and

$$\equiv\, = * u\,\mathbf{0} * \mathbf{0}\,u \wedge = u\,\mathbf{0} = \mathbf{0}\,u \qquad (b)$$

The relations which occur here can be explained by means of the following table:

$= u\,\mathbf{0}$	$= \mathbf{0}\,u$	$\wedge = u\,\mathbf{0} = \mathbf{0}\,u$	$= * u\,\mathbf{0} * \mathbf{0}\,u$	$\wedge = * u\,\mathbf{0} * \mathbf{0}\,u = \mathbf{0}\,\mathbf{0}$	$= * * u\,\mathbf{0}\,\mathbf{0} * * \mathbf{0}$
$\mathbf{V}$	$\mathbf{V}$	$\mathbf{V}$	$\mathbf{V}$	$\mathbf{V}$	$\mathbf{V}$
$\wedge$	$\wedge$	$\wedge$	$\wedge$	$\wedge$	$\wedge$

In passing from the first to the second column assumption (1) is employed, in passing to the third column (*b*) is employed, in passing to the fifth column assumption (2) and the pattern of values of the logical product are employed and in passing to the sixth column (*a*) is employed.

From a consideration of this table it is obvious that $\mathbf{0}$ is the only root of the given equation.

The equation

$$= {*}0{*}00{*}uu$$

will now be examined.

Application of assumption (3) yields:

$$\equiv = {*}0{*}00{*}uu \wedge = 0u = {*}00u$$

If the matrix

u	$=0u$	$={*}00u$	$\wedge=0u={*}00u$	$={*}0{*}00{*}uu$
0	$\mathbf{V}$	$\wedge$	$\wedge$	$\wedge$
${*}EF$	$\wedge$	$\mathbf{V}$ or $\wedge$	$\wedge$	$\wedge$

is examined, it is clear that the equation under consideration has no roots, since all constant expressions, which are not **0**, fall under the pattern ${*}EF$.

It should be observed that an elementary proposition can be equivalent to a semantical equation. For example one can have:

$$\equiv = 00 \equiv = u0 = 0u$$

$$\equiv = {*}000 = {*}0{*}00{*}uu$$

Such equivalences occur when and only when the semantical equation has a constant truth value.

It will be agreed that all constant expressions are roots of an elementary proposition with the value **V** but no constant expression is a root of an elementary proposition with the value **Λ**.

This agreement corresponds to the convention which was established concerning dead substitution.

(*e*) The concept of implication will now be considered.

Equation E is said to imply equation F, if the latter has all the roots of the former.

It is clear that equation F might well have roots which are not roots of equation E. The proposition: *E implies F* will be represented by the symbol $\supset EF$.

If the equations $\mathbf{V} = u0 = {*}00$ and $= u0$ are considered, it is seen that

$$\supset = u0\ \mathbf{V} = u0 = u{*}00$$

will become true, whatever value u may assume, but that this is not the case for

$$\supset \mathbf{V} = u0 = u{*}00 = u0.$$

If the matrix

u	$=u0$	$\mathbf{V}=u0=u{*}00$	$\supset=u0\mathbf{V}=u0=u{*}00$
0	$\mathbf{V}$	$\mathbf{V}$	$\mathbf{V}$
${*}00$	$\wedge$	$\mathbf{V}$	$\mathbf{V}$
${*}{*}000$	$\wedge$	$\wedge$	$\mathbf{V}$

is examined it is apparent that implication is characterized by the following truth values:

If the antecedent is false, the consequent is either true or false. If the antecedent is true the consequent is true.

Because it has been stipulated that all constant expressions are roots of an elementary proposition which has the value **V** any equation implies any elementary proposition which has the value **V**.

Analogously an elementary proposition which has the value **V** implies any semantical equation.

Under these conditions one has

$$\supset E = \mathbf{0\,0}$$
$$\supset = *\,\mathbf{0\,0\,0}\,E$$

The question may now be raised whether the pattern $\supset E\,F$ can be derived from the elementary logical pattern $/\,E\,F$.

The table of values of the pattern $\supset E\,F$ is the following:

E	F	$\supset E\,F$
V	**V**	**V**
V	**Λ**	**Λ**
Λ	**V**	**V**
Λ	**Λ**	**V**

On the other hand the table of values of the pattern $/\,E\,/\,F\,F$ is as follows:

E	F	$/\,E\,/\,F\,F$
V	**V**	**V**
V	**Λ**	**Λ**
Λ	**V**	**V**
Λ	**Λ**	**V**

Obviously the distribution of values in both tables is identical. Consequently the pattern $/\,E\,/\,F\,F$ gives the relation *E implies F*.

Under these conditions a special symbol $\supset E\,F$ need not be introduced. This symbol will, however, be regarded as the abbreviation of the symbol $/\,E\,/\,F\,F$.

4. It is easy to confirm that the calculus of propositions can be constructed without employing the pattern $(\,E\,F\,G\,H\,)$.

It is necessary only to confine oneself to elementary logical functions, i.e. functions which do not contain constant expressions. The pure calculus of propositions so obtained contains only logical letters and the pattern $/\,E\,F$. Every problem of this calculus can be solved by the table or matrix method. In particular it is always possible to determine a logical function of an arbitrary number of variables by accepting values prescribed in advance for given values of the variables.

The following problem can be solved:

To determine the function $\psi(p, q, r)$ which satisfied the conditions:

TABLE I				TABLE II			
p	q	r	$\psi(p, q, r)$	$\mathrm{V} \sim p \mathrm{V} \sim q \sim r$	$\mathrm{V} \sim p \mathrm{V} q \sim r$	$\mathrm{V} p \mathrm{V} \sim q \sim r$	$\mathrm{V} p \mathrm{V} q r$
V	V	V	Λ	Λ	V	V	V
V	V	Λ	V	V	V	V	V
V	Λ	V	Λ	V	Λ	V	V
V	Λ	Λ	V	V	V	V	V
Λ	V	V	Λ	V	V	Λ	V
Λ	V	Λ	V	V	V	V	V
Λ	Λ	Λ	Λ	V	V	V	Λ
Λ	Λ	V	V	V	V	V	V

Table II contains functions each of which has the value **V** for all combinations of values of the letters p, q, and r with the exception of one and in the case of each function the combination is different. The function sought is seen to be $\Lambda \mathrm{V} \sim p \mathrm{V} \sim q \sim r \Lambda \mathrm{V} \sim p \mathrm{V} q \sim r \Lambda \mathrm{V} p \mathrm{V} \sim q \sim r \mathrm{V} p \mathrm{V} q r$.

This method is a simple application of what Hilbert called the normal form for logical expressions.[1] Hilbert showed that every elementary logical function can be represented as the product of sums of logical letters and their negations. This reduction teaches that every theorem of the pure calculus of propositions can be represented as a normal form in which each sum contains both the letter E and the expression $\sim E$.

The matrix calculus as may be seen permits the proof of all the theorems of the calculus of propositions.

5. In conformity with the procedure of Nicod, the French logician who died at an early age, the calculus of propositions will be developed from two axioms, the first of which is called the *syllogism of Nicod* [2] and the second the *principle of logical identity*.[3] These axioms are the following:

(1) $\supset / p / q r \supset / s q / p s,$

(2) $\supset p p.$

Nicod showed that these two axioms can be reduced to one. This interesting result will not be employed because it is desired that needless complication of the calculus be avoided and because axiom (2) follows from certain axioms which

[1] Cf. David Hilbert und W. Ackermann: *Grundzüge der theoretischen Logik*, II Aufl., Berlin, 1938, p. 10.

[2] Nicod: *l. c.*, p. 38.

[3] Nicod: *l. c.*, p. 34.

must be accepted in the complete system of elementary semantics.

In addition a *rule of substitution* which permits one to substitute letters or any elementary logical functions for letters will be accepted here just as it was accepted by Nicod. The *rule of modus ponens*, i.e. *If E and* $\supset EF$ *are theorems, then F is a theorem*, will also be accepted.

It will now be seen how these axioms and rules are employed.

If p is substituted for q and r in (1), and (2) applied,

$$\supset \supset pp \supset / sp / ps$$

will be obtained, from which follows on the application of the rule of modus ponens the so-called principle of permutation :

(3) $$\supset / sp / ps.$$

If now $\sim s$ is substituted for s and $\sim p$ for p, a variant of this principle :

(3*a*) $$\supset \vee sp \vee ps,$$

will be obtained.

If the principle of permutation is applied to the principle of identity,

(4) $$/ \sim tt$$

will be derived.

If $\sim t$ is substituted for t and the result abbreviated,

(5) $$\supset \sim \sim tt$$

will result ; this is the first principle of double negation.

Since (5) is a theorem, if $\sim \sim t$ is substituted for p and t for q and r in (1) and the rule of modus ponens applied to the result

$$\supset / st / \sim \sim ts$$

will be obtained.

Since (2) is a theorem, if $\sim s$ is substituted for t and the rule of modus ponens applied to the result

$$/ \sim \sim \sim ss$$

will be derived.

If (3) is applied and the result abbreviated, the second principle of double negation :

(6) $$\supset s \sim \sim s$$

will be obtained.

It should now be noted that principle (5) yields the so-called principle of tautology :

(7) $$\supset \vee ppp$$

if appropriate abbreviations are made.

If q is substituted for r in (1) the principle :

(1*a*) $$\supset \supset pq \supset / sq / ps$$

will be obtained.

In practice this principle permits the use of the rule of the syllogism.

If it is supposed that the theorems: $\supset E\,F$ and $\supset F\,G$ have been proved the latter theorem may be transformed in the same way as the principle of identity was transformed into theorem (4). In other words if the principle of permutation is applied, the theorem $/ \sim G\,F$ will be obtained. If then E is substituted for p, F for q, and $\sim G$ for s in (1a)

$$\supset \supset E\,F \supset / \sim G\,F\,/\,E \sim G,$$

will be obtained.

If the theorems $\supset E\,F$ and $/ \sim G\,F$ are asserted and the rule of modus ponens applied twice, the theorem $/\,E \sim G$ will result. This theorem when abbreviated has the form $\supset E\,G$.

It is therefore clear that the theorem $\supset E\,G$ can always be proved if the theorems $\supset E\,F$ and $\supset F\,G$ can be proved.

If $\sim s$ is substituted for s in (1a), the principle of permutation applied and the result abbreviated, the principle of the syllogism:

(8) $$\supset \supset p\,q \supset \supset q\,s \supset p\,s$$

will be obtained.

In all further proofs obviously the usual rule of the syllogism can in practice be employed as an abbreviation.

From this principle the so-called principle of conversion may be derived without difficulty.

If after $\sim \sim q$ is substituted for p the rule of modus ponens is applied to (8) and to (5).

$$\supset \supset q\,s\,/ \sim \sim q \sim s$$

will be obtained.

On the other hand application of the principle of permutation yields:

$$\supset / \sim \sim q \sim s\,/ \sim s \sim \sim q.$$

If now the rule of the syllogism is applied to the last two theorems and the result abbreviated, the principle of conversion

(9) $$\supset \supset q\,s \supset \sim s \sim q$$

will be obtained.

If $\sim s$ is substituted for p, $\sim q$ for q, and r for s in the principle of the syllogism,

$$\supset \supset \sim s \sim q \supset \supset \sim q\,r \supset \sim s\,r$$

will result. When abbreviated the latter theorem has the form:

$$\supset \supset \sim s \sim q \supset \vee\, q\,r \vee s\,r.$$

If the rule of the syllogism is applied to (9) and to the last theorem, the principle of summation:

(10) $\supset \supset q s \supset \vee q r \vee s r$

will result.

To obtain the so-called principle of addition first the principle of permutation is written in the following form:

$$/ / r q \sim / q r$$

If the principle of permutation is applied to this theorem:

$$/ \sim / q r / r q$$

will result.

If in axiom (1) $\sim / q r$ is substituted for p, r, and q and q for r, and if the rule of modus ponens is applied to the result.

$$\supset / s r / \sim / q r s$$

will be obtained.

If $\sim s$ is substituted for r in the last theorem and the rule of modus ponens applied to (2),

$$/ \sim / q \sim s s$$

will result.

If the principle of permutation is applied to this theorem the result is:

$$/ s \sim / q \sim s,$$

which may be written in the alternative form

(11) $\supset s \supset q s.$

If $\sim q$ is substituted, for q in (11) and the result abbreviated, the principle of addition:

(12) $\supset s \vee q s$

will be obtained.

The principle of associativity can easily be derived by employing the principles of addition and summation, the rule of modus ponens and the rule of the syllogism. If after making the necessary substitutions the principle of summation is combined with the principle of addition and then to the result of this operation, the theorem:

$$\supset \vee \vee p r q \vee \vee \vee p q r q$$

will be obtained.

On the other hand on the application of the principles of permutation, addition, and the rule of the syllogism to the principle of addition:

$$\supset q \vee \vee p q r$$

will be obtained.

If now the principle of summation is applied to this theorem

$$\supset \vee \vee \vee p q r q \vee \vee \vee p q r \vee \vee p q r$$

will result.

By the application of the principle of tautology and of the rule of the syllogism the theorem:

$$\supset \vee \vee \vee p q r q \vee \vee p q r$$

will result.

The application of the principle of the syllogism to these two theorems yields

(13) $$\supset \vee \vee p r q \vee \vee p q r$$

This is the principle of associativity.

This proof was given independently by Bernays [1] and Łukasiewicz.[2]

The theorem which is necessary for the construction of Hilbert's normal form must still be derived.

The derivation begins with theorems

(14) $$\supset \supset p q \vee \sim p q$$

(15) $$\supset \vee \sim p q \supset p q,$$

whose demonstration depends upon the application of the principles of double negation.

These theorems permit the derivation of theorem

(16) $$\supset p \supset q \wedge p q$$

which is obtained by first substituting $\wedge p q$ for p in the principle of identity and then employing (14), the principle of double negation, the principle of associativity, and finally principle (15).

Where $\wedge p q$ is substituted for s in the principle of summation and principle (14) is applied, the theorem

(17) $$\supset \supset q \wedge p q \vee \sim \vee q r \vee \wedge p q r$$

will be obtained.

If the principle of the syllogism is applied to (16) and (17)

(18) $$\supset p \vee \sim \vee q r \vee \wedge p q r$$

will result.

The application of the principle of summation to theorem (18) yields

(19) $$\supset \vee p r \vee \vee \sim \vee q r \vee \wedge p q r r.$$

By following the procedure used in proving the principle of associativity and applying this principle

(20) $$\supset \vee \vee \sim \vee q r \vee \wedge p q r r \vee \sim \vee q r \vee \vee \wedge p q r \vee \wedge p q r$$

will be obtained.

The principles of tautology and summation yield

(21) $$\supset \vee \sim \vee q r \vee \vee \wedge p q r \vee \wedge p q r \vee \sim \vee q r \vee \wedge p q r.$$

If the principle of the syllogism is applied to (19) and (20) and then to the result of this operation and to (21)

[1] P. Bernays: "Axiomatische Untersuchung des Aussagen-Kalküls der *Principia Mathematica*," *Mathematische Zeitschrift*, Bd. 25, 1926, pp. 312–13.

[2] Jan Łukasiewicz: *Elementy logiki matematycznej* (*Elements of Mathematical Logic*), *l.c.*, p. 87.

(22) $\supset \vee pr \vee \sim \vee qr \vee \wedge pqr$

will result.

The application of (15) to this theorem yields

(23) $\supset \vee pr \supset \vee qr \vee \wedge pqr.$

If (14), the principle of associativity and the principles of double negation are successively applied, theorem

(24) $\supset \vee \wedge pr \vee qr \vee \wedge pqr$

will follow from (22).

Application of the principle of conversion to (24) yields

(25) $\supset \wedge \vee pqr \vee \wedge pr \wedge qr.$

Theorems (24) and (25) constitute the basis of Hilbert's reduction. It must still be shown that in the implications involved in these theorems the role of the consequent and the antecedent can be interchanged.

By applying the principle of permutation and theorem (15) theorem

(26) $\supset \supset rp \supset \supset rq \supset r \wedge pq$

follows from (23).

In this theorem $\vee \wedge pqr$ is to be substituted for r, $\vee pr$ for p, and $\vee qr$ for q and the following theorems:

$$\supset \vee \wedge pqr \vee pr$$
$$\supset \vee \wedge pqr \vee qr,$$

which are obtained by applying the principle of conversion to the principle of addition and then using first the principles of double negation and then the principle of summation, are to be employed.

This method permits the double application of the rule of modus ponens so that finally the theorem

(27) $\supset \vee \wedge pqr \wedge \vee pr \vee qr$

will be obtained.

By applying the principle of conversion to (27)

(28) $\supset \vee \wedge pr \wedge qr \wedge \vee pqr$

will be obtained.

The interesting part of the calculus of propositions terminates with these theorems. All the other theorems of the calculus can be proved by reducing them to the normal form of Hilbert. A long series of tedious operations is required to achieve this end, but no real difficulties are presented.

It can easily be shown by the use of Hilbert's method that the elementary calculus of propositions which has been presented yields all the results which can be obtained by means of the matrix method and no others. From this point of view the consideration of Hilbert's method is of no great intrinsic interest, but it must be considered in the development of the complete system of elementary semantics.

6. The logic of Aristotle was based upon the propositional patterns: *All S are P*, (*S* **a** *P*), *No S is P* (*S* **e** *P*) *Some S are P* (*S* **i** *P*), *and Some S are not P* (*S* **o** *P*).[1]

Aristotle sought the relations between these patterns and it must be acknowledged that with the exception of the fundamental mistake which he made in regarding the inference of $S\,\mathbf{i}\,P$ from $S\,\mathbf{a}\,P$ as valid, he solved his problem in an impressive way. Nevertheless his choice of these particular patterns was connected with the linguistic tradition of his day and the question whether the fundamental relations between these patterns are independent or whether they can be simplified was left open. Unfortunately the relations were presented in such a way that mankind thrust the problem aside and did not thoroughly consider it until recent times. Even to-day logic texts are employed which contain the logical square and the erroneous theorem that $S\,\mathbf{i}\,P$ is a consequence of $S\,\mathbf{a}\,P$. Even to-day the ontological pattern: *A is B* is employed despite its fundamental ambiguity. Finally the Eulerian method of comparing the extension of concepts by means of circumlocutions is still employed although it fails in even the simplest cases.

The critical analysis of the patterns of Aristotle is due to the great contemporary English logician, Bertrand Russell.[2]

Russell's analysis led to the reduction of the patterns of Aristotle to the pattern: x **is** B where x denotes any individual, the concept for all x, which is abbreviated by $\mathbf{\Pi}\,\bar{x}$ and which is called a quantifier, and the concepts of the elementary calculus of propositions.

Russell's analysis may be summarized by means of the following table:

$$\begin{array}{r|l}
S\,\mathbf{a}\,P & \mathbf{\Pi}\,\bar{x} \supset (\bar{x}\ \mathbf{is}\ S)(\bar{x}\ \mathbf{is}\ P) \\
S\,\mathbf{e}\,P & \mathbf{\Pi}\,\bar{x} \supset (\bar{x}\ \mathbf{is}\ S)\sim(\bar{x}\ \mathbf{is}\ P) \\
S\,\mathbf{i}\,P & \sim\mathbf{\Pi}\,\bar{x} \supset (\bar{x}\ \mathbf{is}\ S)\sim(\bar{x}\ \mathbf{is}\ P) \\
S\,\mathbf{o}\,P & \sim\mathbf{\Pi}\,\bar{x} \supset (\bar{x}\ \mathbf{is}\ S)(\bar{x}\ \mathbf{is}\ P)
\end{array}$$

The proposition: *All men are mortal* for example reduces to the proposition: *All individuals who are men are mortal.* The proposition: *No man is mortal* is replaced by the proposition *All men are immortal.* The proposition *Some men are*

[1] Cf. Aristotle: *Analytica Priora*, translated by A. J. Jenkinson, Oxford, 1928, Bk. 1, 2, 25a.

[2] Cf. Bertrand Russell and Whitehead · *Principia Mathematica*, 1st ed., Cambridge, 1910, vol. 1, p. 21.

mortal is the contradictory of the proposition: *No man is mortal* and the proposition: *Some men are not mortal* is the contradictory of the proposition: *All men are mortal.*

If now the abbreviation $\exists\, \bar{x}\, F$ is introduced for propositions of the form $\sim \Pi\, \bar{x} \sim F$, after the usual transformations of the calculus of propositions have been made one obtains

$\exists\, \bar{x} \wedge (\bar{x} \text{ is } S)\,(\bar{x} \text{ is } P)$ instead of S i P and
$\exists\, \bar{x} \wedge (\bar{x} \text{ is } S) \sim (\bar{x} \text{ is } P)$ instead of S o P.

To say that not all individuals satisfy the condition $\sim F$ is, in conformity with the principle of excluded middle, to say that some individuals satisfy the condition F, or that there exist individuals which satisfy the condition F.

Thus the calculus of propositions leads to the conclusion that the contradictory of the proposition: *No man is mortal* is equivalent to the proposition: *There exist individuals who are men and who are mortal.* The contradictory of the proposition: *All men are mortal* is equivalent to the proposition: *There exist individuals who are men and who are not mortal.* These results are consistent with primitive intuition.

The comparison of the patterns:

$\Pi\, \bar{x} \supset (\bar{x} \text{ is } S)\,(\bar{x} \text{ is } P)$ and
$\sim \Pi\, \bar{x} \supset (\bar{x} \text{ is } S) \sim (\bar{x} \text{ is } P)$

which replace the patterns S a P and S i P respectively lead to the conclusion that formal implication between the first and the second patterns is not valid. This result is compatible with the intuitively confirmed thesis that it is invalid to infer S i P from S a P in the logical square.

On the basis of this analysis Russell reduced Aristotle's theory of the syllogism to the elementary calculus of propositions and the calculus of propositional functions.

The late Professor Leśniewski [1] opposed the *calculus of names* to the method of analysis given by Russell. The two methods differ in so far as the former is based not upon the general pattern of a propositional function but upon the pattern *A is B*, where *A* is not an element of the class *B* but the class whose only element is a certain element of the class *B*.

This modification is justifiable only because of Leśniewski's desire to employ the traditional pattern *A is B*. It does not, however, yield anything essentially new. One can hardly agree with Professor Kotarbiński that Leśniewski's

[1] Stanisław Leśniewski: "Über die Grundlagen der Ontologie," *Comptes rendus des séances de la société des sciences et des lettres de Varsovie*, Classe III, Année 23, 1930, pp. 111–132.

> "... system of the calculus of names is the most mature, natural, and practical ... in its applications of all the systems known to us ".[1]

On the contrary it seems to me that this system is artificial and complicated. However, I agree with Kotarbiński that Leśniewski's calculus of names

> "... is very closely related to the traditional Aristotelian formal logic ".[2]

but this is no great recommendation. A calculus which is based upon the vague concept of a name and which postulates the existence of only one designate of a name is clearly metaphysical and consequently cannot be employed as a constituent element of the system of the mathematical sciences.

7. On the level of everyday language it is possible to speak only of rough orientation. Everyday language contains no universal propositions with precisely determined meanings because it has no precise substitution rules.

Quite otherwise is the case in the sphere of operation of the rules of the system of elementary semantics. If, for example, the proposition: $\mathbf{\Pi}\bar{x}(\bar{x}\mathbf{0}\mathbf{0}\bar{x})$ is constructed it is plain that this proposition by itself has no clear meaning. However, within the system of elementary semantics it can be completely determined if precise construction rules for such propositions and for theorems in which such propositions appear, are accepted. This is the fundamental difference between intuitive and symbolic logic. Intuitive logic in itself is but roughly determined. However, certain simple rules which permit the construction of precisely determined expressions and theorems can be derived from it. Thus the logic of universal propositions can be obtained automatically without employing any universal propositions in its constructions in the following manner:

The calculus of quantifiers is usually constructed for patterns of propositional functions such as $\Phi\{x\}$, $\Psi\{x\}$, $f(x, y)$, $g(x, y)$, etc.

To accept such patterns a special act of intuition is required. Consequently the foundations of logic becomes a domain which is difficult to understand. Later it will be seen that these patterns can be constructed with the help of the concepts of semantics. On the other hand it is easy to show that the

[1] Kotarbiński: *Elementy teorji poznania, logiki formalnej i metodologji nauk* (*Elements of the Theory of Knowledge, Formal Logic and the Methodology of the Sciences*), Lwów, 1929, p. 253.

[2] Kotarbiński: *l.c.*

calculus of quantifiers can be constructed without employing these patterns.

Semantical functions alone will be employed. The semantical letters $\bar{x}$, $\bar{y}$, $\bar{z}$ . . . may be employed as apparent semantical variables.

The following rule will also be accepted:

If E is a semantical function which contains the semantical letter I and which does not contain the apparent semantical variable K, and if F is the result of the substitution of K for I in E, then $\mathbf{\Pi}\, K\, F$ *is a logical expression.*

Consequently the following expressions are logical expressions:

$\mathbf{\Pi}\,\bar{x}\,(\,\bar{x}\,\mathbf{0}\,\mathbf{0}\,\bar{x}\,)$, $\mathbf{\Pi}\,\bar{x}\,\mathbf{\Pi}\,\bar{y}\,(\,\bar{x}\,\bar{y}\,\bar{x}\,\bar{y}\,)$, $\sim\mathbf{\Pi}\,\bar{x}\sim\,=\,\mathbf{0}\,\bar{x}$, etc.

The pattern $\sim\mathbf{\Pi}\, K \sim F$ will be abbreviated by the symbol

$$\exists\, K\, F.$$

The expression $\mathbf{\Pi}\; K\, F$ is to be read:

for all K, F occurs.

The expression $\sim\mathbf{\Pi}\, K \sim F$ is to be read:

there exists a K such that F, or for certain K's, F occurs.

The whole calculus of quantifiers reduces to the following rule of procedure which is called the *rule of generalization.*

If $$\wedge \supset L\, M \supset M\, N$$
is a theorem, if M does not contain the apparent variable K, and contains the semantical letter I which is not contained in L and if finally F is the result of the substitution of K for I in M, then

$$\wedge \supset L\, \mathbf{\Pi}\, K\, F \supset \mathbf{\Pi}\, K\, F\, N$$

is a theorem.

The proof of the theorem:

$$\supset \mathbf{\Pi}\,\bar{x}\,\{\,\bar{x}\, v\,\}\,\{\,u\, v\,\}$$

will now be developed with the help of this rule.

$$\wedge \supset \wedge\, p \sim p\,\{\,u\, v\,\} \supset \{\,u\, v\,\}\,\{\,u\, v\,\}$$

is a theorem.

If $\wedge\, p \sim p$ is substituted for L, $\{\,u\, v\,\}$ for M, and $\{\,u\, v\,\}$ for N in the rule which has just been given, M obviously does not contain the apparent variable $\bar{x}$ and does contain the semantical letter u which is not contained in L, and finally $\{\,\bar{x}\, v\,\}$ is the result of the substitution of $\bar{x}$ for u in M. Consequently the application of the given rule yields the theorem:

$$\wedge \supset \wedge\, p \sim p\, \mathbf{\Pi}\,\bar{x}\,\{\,\bar{x}\, v\,\} \supset \mathbf{\Pi}\,\bar{x}\,\{\,\bar{x}\, v\,\}\,\{\,u\, v\,\}.$$

The desired theorem follows immediately when simple operations of the elementary calculus of propositions are performed.

It is obvious that the latter theorem is a special case of the so-called axiom of deduction, which can be formulated in terms of the functional pattern $\Phi\{x\}$ as follows:

$$\supset \Pi \bar{x}\, \Phi\{\bar{x}\}\, \Phi\{u\}$$

This formulation of the axiom is possible only if functional patterns are accepted. However, for purposes of orientation it can be employed as a provisional abbreviation. This axiom can be proved by the same method as that employed above if the pattern $\Phi\{u\}$ is substituted for the semantical function $\{u\,v\}$.

The principle of disjunction will now be developed for any semantical function $\Phi\{u\}$.

The application of the principle of deduction to the function $\mathsf{V}\, p\, \Phi\{u\}$ yields the theorem

$$\supset \Pi \bar{x}\, \mathsf{V}\, p\, \Phi\{\bar{x}\}\, \mathsf{V}\, p\, \Phi\{u\}.$$

Simple operations of the elementary calculus of propositions transform this theorem into the theorem:

$$\supset \Lambda\, \Pi \bar{x}\, \mathsf{V}\, p\, \Phi\{\bar{x}\} \sim p\, \Phi\{u\}.$$

In order to apply the rule which was given above, the theorem:

$$\Lambda \supset \Lambda\, \Pi \bar{x}\, \mathsf{V}\, p\, \Phi\{\bar{x}\} \sim p\, \Phi\{u\} \supset \Phi\{u\}\, \Phi\{u\}$$

must now be proved.

$\Lambda\, \Pi \bar{x}\, \mathsf{V}\, p\, \Phi\{\bar{x}\} \sim p$ will be substituted for L, $\Phi\{u\}$ for M and for N. It is clear that the logical expression L cannot contain the semantical letter u because $\Phi\{\bar{x}\}$ was derived from $\Phi\{u\}$ by substituting $\bar{x}$ for u. The moment $\Pi \bar{x}\, \mathsf{V}\, p\, \Phi\{\bar{x}\}$ was accepted as a logical expression, it was supposed that $\Phi\{u\}$ does not contain the apparent variable $\bar{x}$. Otherwise the expression $\Pi \bar{x}\, \mathsf{V}\, p\, \Phi\{\bar{x}\}$ could not be constructed. The conditions of the rules are therefore satisfied and the theorem:

$$\Lambda \supset \Lambda\, \Pi \bar{x}\, \mathsf{V}\, p\, \Phi\{\bar{x}\} \sim p\, \Pi \bar{x}\, \Phi\{\bar{x}\} \supset \Pi \bar{x}\, \Phi\{\bar{x}\}\, \Phi\{u\}$$

can be obtained.

From this theorem it is easy to pass to the so-called principle of disjunction:

$$\supset \Pi \bar{x}\, \mathsf{V}\, p\, \Phi\{\bar{x}\}\, \mathsf{V}\, p\, \Pi \bar{x}\, \Phi\{\bar{x}\}$$

Like the principle of deduction the principle of disjunction is a metalogical pattern which tells how the principle of disjunction is to be proved for any given semantical function. If for example it is desired that this principle be proved for the semantical function $\{u\,v\}$ already considered, it would be necessary to repeat the above proof, substituting the function $\{u\,v\}$ for the pattern $\Phi\{u\}$. In practice this is obviously unnecessary since the pattern of the principle of disjunction can be employed.

It will now be seen that the rule which has been given permits the generalization of theorems. If it is supposed that the theorem $\Phi\{u\}$ has been proved, the theorem

$$\Lambda \supset \supset p\, p\, \Phi\{u\} \supset \Phi\{u\}\, \Phi\{u\}$$

can certainly be proved.

The application of the rule of generalization to this theorem yields the theorem :

$$\Lambda \supset \supset p\, p\, \Pi \bar{x}\, \Phi\{\bar{x}\} \supset \Pi \bar{x}\, \Phi\{\bar{x}\}\, \Phi\{u\}$$

provided that $\Phi\{u\}$ does not contain the apparent variable $\bar{x}$. If it does contain $\bar{x}$ obviously another variable will be employed.

The theorem $\Pi \bar{x}\, \Phi\{\bar{x}\}$ is derived from the last theorem by performing simple operations of the elementary calculus of propositions.

This method permits the derivation of various theorems of the calculus of quantifiers in a simple manner.

For example to derive the syllogism Barbara first the theorem

$$\supset \Lambda\, \Pi \bar{x} \supset \phi\{\bar{x}\}\, \psi\{\bar{x}\}\, \Pi \bar{x} \supset \psi\{\bar{x}\}\, \sigma\{\bar{x}\}\, \Lambda \supset \phi\{u\}\, \psi\{u\} \supset \psi\{u\}\, \sigma\{u\}$$

will be proved, by applying the principle of deduction separately to the functions

$$\supset \phi\{u\}\, \psi\{u\} \text{ and } \supset \psi\{u\}\, \sigma\{u\}.$$

This theorem leads to the theorem

$$\supset \Lambda\, \Pi \bar{x} \supset \phi\{\bar{x}\}\, \psi\{\bar{x}\}\, \Pi \bar{x} \supset \psi\{\bar{x}\}\, \sigma\{\bar{x}\} \supset \phi\{u\}\, \sigma\{u\}.$$

The application of the principle of generalization to the last theorem immediately yields the theorem

$$\supset \Lambda\, \Pi \bar{x} \supset \phi\{\bar{x}\}\, \psi\{\bar{x}\}\, \Pi \bar{x} \supset \psi\{\bar{x}\}\, \sigma\{\bar{x}\}\, \Pi \bar{x} \supset \phi\{\bar{x}\}\, \sigma\{\bar{x}\}$$

which is the symbolic transcription of the syllogism Barbara.

8. In addition to two-valued logic there exist various. systems of many-valued logic.

The oldest is the system of Professor N. A. Vasiliev who published his work between 1910 and 1913. Vasiliev mentioned these works briefly at the meeting of the International Congress of Philosophy which was held in Naples.[1]

Vasiliev accepts not only the propositions : *S is P* and *S is non-P* but also the proposition *S is P* and *non-P*. He constructs a consistent system on the basis of these suppositions. Clearly such propositions as *S is P* and *non-P* are undecidable

[1] Cf. N. A. Vasiliev : "Imaginary (non-Aristotelian) Logic," *Atti del Quinto Congresso Internazionale di Filosofia*, Napoli 5–9 Maggio, 1924, pp. 107–109.

propositions. For example it can be said of some things that they are white and not-white; it can be said that electrons are real and not-real.

Such propositions play a large role in actual life and are undoubtedly very interesting because they lead one beyond the sphere of exact thinking. It is difficult to predict whether interesting results will be obtained if these phenomena are included within a system. The supposition that concepts have a determinate meaning is the basis of efforts of this kind. But actually the propositions: this is white and not white, and electrons are real and not real merely confirm that the concepts involved have no precisely determined sphere.

Vasiliev's theory was directed against conceiving the principle of contradiction in too general a fashion.

Professor Łukasiewicz went even further than Vasiliev in this direction. He was of the opinion that if in the exact sciences the existence of contradiction were demonstrated, the contradiction should be accepted as a valuable result.[1]

However, it must be acknowledged that Łukasiewicz soon abandoned this unfortunate position and devoted his complete attention to another matter, namely the discussion of *possible propositions*. Łukasiewicz is rightly regarded as the highest authority to-day on questions concerning the calculus of propositions and he has actually obtained far reaching results in this field. His conception of a many-valued logic, which contains $\frac{1}{2}$ true and $\frac{1}{4}$ true propositions, etc., as well as true and false propositions is undoubtedly worthy of consideration. He advances ideas akin to those of Brouwer,[2] Post,[3] and C. I. Lewis,[4] but presents his views with far greater clarity and simplicity.

Łukasiewicz appeals to the fact that while Aristotle was familiar with the principle of excluded middle he did not accept it without reservations because it is not applicable to propositions which refer to future contingent events. The

[1] Jan Łukasiewicz: *O zasadzie sprzeczności u Arystotelesa. Studyum Krytyczne* (*On the Principle of Contradiction in Aristotle. A Critical Study*), Kraków, 1910, p. 151.

[2] L. E. J. Brouwer: "Begründung der Mengenlehre unabhängig vom logischen Satz von ausgeschlossen Dritten, I." *Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam*, Eerste sectie, vol. 12, no. 5, Amsterdam, 1918.

[3] E. L. Post: "Introduction to a General Theory of Elementary Propositions," *American Journal of Mathematica*, vol. 43, 1921, pp. 163–185

[4] C. I. Lewis: "Note Concerning Many-valued Logical Systems," *Journal of Philosophy*, vol. 30, 1933, p. 364.

Epicureans did not accept this principle. Its actual creator was Chrysippus, a founder of the Stoic school.[1]

If it is desired to account for possible propositions, the principle of excluded middle must be regarded as false.

If for example, following Łukasiewicz, the proposition:

> "I will be in Warsaw next year on the 21st of December at noon,"[2]

is considered, it must be admitted that this proposition is neither true nor false. Neither the value 1, which corresponds to the value true, nor the value 0 which corresponds to the value false, can be assigned to it. Its truth value can only be $\frac{1}{2}$.

Clearly the signs 1 and 0 correspond to the signs **V** and **Λ** which were introduced above.

On the basis of these remarks, Łukasiewicz develops the calculus of propositions in a three-valued logic. He then passes to a many-valued logic.[3] I will confine myself here to the presentation of a matrix which explains the values of the patterns $\supset E\ F$ and $\sim F$ in all possible cases. I will use the signs 1, 0, and $\frac{1}{2}$.

E	F	$\sim E$	$\supset E\ F$
1	1	0	1
1	$\frac{1}{2}$	0	$\frac{1}{2}$
1	0	0	0
$\frac{1}{2}$	1	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
0	1	1	1
0	$\frac{1}{2}$	1	1
0	0	1	1

The calculus itself is interesting, but several serious questions must be considered. In the first place it is clear that many restrictions are imposed upon the structure of a system of mathematics and it is dubious whether the calculus can do without a theory of types. In other words it is doubtful whether a many-valued logic is of any use in constructing a

[1] Cf. J. Lukasiewicz · "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalkuls," *Comptes rendus des séances de la société des sciences et des lettres de Varsovie*, Classe III, année 23, 1930, p. 63.

[2] Łukasiewicz: *l c.*, p. 64.

[3] Cf. Jan Łukasiewicz and Alfred Tarski: "Untersuchungen uber den Aussagenkalkül," *Comptes rendus*, etc., *l.c.*, PP. 3, pp. 38–42.

complete system of mathematics and metamathematics. On the other hand since the calculus of probability can be developed within the framework of two-valued logic it is difficult to see what advantage might be derived from the use of the calculus of Łukasiewicz.

The philosophical arguments which Łukasiewicz employs and in particular the Aristotelian classification of events into contingent and necessary are quite naïve and antagonize the reader.

The question whether or not my stay in a given place at a definite time is a matter of chance cannot be decided as simply as Mr. Łukasiewicz seems to think. It suffices to point out that unknowingly I might have swallowed poison early this morning and consequently I will die within a few hours. If this is the case it is certain that I will be unable to meet my classes next year.

In any case a thorough reconsideration of all these matters is necessary.

Chapter VI

THE THEORY OF CLASSES

1. The calculus of classes developed from investigations concerning the comparison of the domains of concepts. Classes are denoted by the letters A, B, C . . . and symbolic relations between these letters are introduced. The symbolism of Peano [1] will be employed, but to avoid parentheses the symbol will be written first followed by the letters, in conformity with the suggestion of Łukasiewicz.[2]

The equality of two classes A and B will be denoted by the symbol $\underset{\mathbf{Cl}}{=} A\ B$. The sign $\underset{\mathbf{Cl}}{=}$ denotes class equality and is to be distinguished from semantical identity. Classes are equal if they have the same elements. However, this does not imply that the classes are identical. For example, when one has $\underset{\mathbf{Cl}}{=}$ **(man) (featherless biped)**, the concepts **man** and **featherless biped** are not identical. The notion of the identity of two equal classes has raised problems which have long troubled logicians. Recently the so-called *axiom of extensionality* which assures the identity of two equal classes [3] has been generally accepted. This axiom has proved to be superfluous because one can confine oneself to extensional classes.[4]

If the elements of a class A are at the same time elements of class B one writes $\subset A\ B$, and it is said that *A is a subclass of the class B*. If A is a class $-\ A$ is the class of objects which are not elements of class A. If A and B are classes $\cap\ A\ B$ is the class of objects which are elements of the classes A and B, $\cup\ A\ B$ is the class of objects which are elements of the class A or the class B. The class $\cap\ A\ B$ is called the product of the classes A and B, the class, $\cup\ A\ B$ is called the sum of the classes A and B.

The first analysis of the relation $\subset A\ B$ was made by Leibniz [5]

[1] Giuseppe Peano: *Formulaire de Mathématiques*, Turin, 1895, Tome I.

[2] Jan Łukasiewicz: *Elementy logiki matematycznej* (*Elements of Mathematical Logic*) (lithographed), Warszawa, 1929, p. 40.

[3] Rudolf Carnap: *Der logische Aufbau der Welt*, Berlin, 1928, pp 59 ff.

[4] Cf. *T. C. T.*, p. 42 f.

[5] *Die philosophischen Schriften* von Gottfried Wilhem Leibniz, Herausgegeben von C. J. Gerhardt, Bd. VII, Berlin, 1890, p. 223.

who proved among other things the following theorem, which he called the *praeclarum theorema*:

If $\subset A\,B$ and $\subset C\,D$, then $\subset \cap\, A\,C \cap B\,D$.

The modesty of this result is striking when it is contrasted with the great ideal of Leibniz.

Sleszyński writes the following concerning this matter:

> "Leibniz seeks a logic which will be the instrument of investigation and which will lead to the discovery of new truths as well as to the proof of known truths. He developed a logic with a form very different from the one about which he dreamt. Thus mathematical logic even in its earliest days took the form of the algebra of logic or the logic of classes. Although this logic has interesting and important applications, for example in the calculus of probability, the number of such applications is small. A comparison of the gigantic mathematical discoveries of Leibniz with his meagre achievements in logic reveals the difficulties involved in the development of logic. It is apparent that the foundation of logic is rocky and barren and it is difficult to make any real progress in it. However, Leibniz himself showed by his activities how very useful to other sciences studies in this sphere may be." [1]

Leibniz' disciples, Segner and Lambert, introduced the signs $>$ and $<$ to denote the relation $\subset A\,B$.

Segner wrote: *animal* $>$ *vertebrate* to indicate that the denotation of the concept *animal* is greater than the denotation of the concept *vertebrate*. Lambert wrote *animal* $<$ *vertebrate* to denote that the connotation of the concept *vertebrate* is richer than the connotation of the concept *animal*.[2]

In 1871 Lambert discovered the distributive law of the sign $\cap$ with respect to the sign $\cup$.[3] The relevant theorem can be formulated as follows:

$$\underset{\mathrm{Cl}}{=} \cap\, C \cup A\,B \cup \cap\, C\,A \cap C\,B.$$

Lambert also proved a number of other theorems of the calculus of classes, but this was only the beginning. Lambert and Segner had no immediate successors and exerted no direct influence on the further development of the calculus of classes.

Euler's [4] method, which involves the comparison of domains of concepts and which utilizes circles to represent these

[1] Jan Sleszyński: *Teorja dowodu* (*The Theory of Proof*), Tom II, Kraków, 1929, p. 4.

[2] Cf. Alessandro Padoa. "La logique déductive dans sa dernière phase de développement," *Revue de Métaphysique et de Morale, année* 19, 1911, p. 848.

[3] Padoa: *l.c.* Année 20, 1912, p. 58.

[4] Cf. Sleszyński. *l.c.*, Tom I, 1925, p. 65.

domains became very popular, and despite its inaccuracy is employed to this very day in textbooks. To become convinced of its inaccuracy, the comparison of the concepts non-Pole and non-German should be attempted. This is a question a clever schoolboy might well ask his teacher. In as much as two circles whose areas represent the extension or connotation of the two concepts cannot be drawn, these concepts cannot be compared. If the area of the smaller circle represents the extension of the concept Germans, and the area of the larger circle represents the extension of the concept non-Poles, the whole plane outside the smaller circle represents the non-Germans. Consequently the plane outside the circles as well as the area of the circles themselves must be considered. It should be observed that even here it is not sufficient to employ circles and that a closed plane must be used.

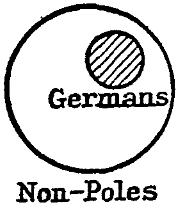

FIG. 4.

The concepts (A): *a man with at least one son*, (B): *a man with at least one daughter*, (C): *a man with one child* cannot be compared even if both circles and the portions of the planes exterior to them are employed. However, they can be compared if polygons are used. In Figure 5 the polygon $H N P Q S L$ represents the extension of (A), the polygon $I M O Q S K$ that of (B), and the square $O P T R$ that of (C). The square $M N O P$ which is common to the two polygons representing the extension of (A) and (B), represents the men with at least one son and one daughter. But these supplementations cannot save Euler's method, since the latter is not applicable to null classes and consequently presents a false representation of classes, which is based upon the assumption that a class is a material collection.

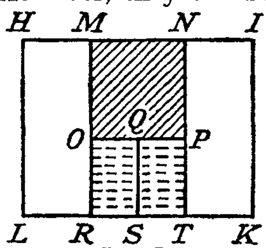

FIG. 5.

In the middle of the nineteenth century a new series of attempts to construct the calculus of classes appeared.

In 1858 De Morgan proved the following theorems:

$$\underset{\mathrm{Cl}}{=} - \cup A B \cap - A - B$$

$$\underset{\mathrm{Cl}}{=} - \cap A B \cup - A - B$$ [1]

[1] Padoa: *l.c.*, année 20, 1912, p. 207.

which even to-day bear his name. It is difficult for the layman to understand these theorems. The following examples illustrate them :

To say : *it is not true, that this man is either stupid or bad,* is to say : *this man is not-stupid and not-bad.* To say : *it is not true that this man is stupid and bad,* is to say : *this man is either not-stupid or not-bad.*

In 1854 George Boole [1] discovered the associative property of the sign $\frown$. Not until 1877 was the associative property of the sign $\smile$ discovered by Schröder.[2]

The concept of the *universal* class and that of the *null* class were introduced by Boole,[3] who denoted them by the numbers 1 and 0 respectively. A universal class contains as elements all the objects about which we are speaking. For example, in arithmetic the class of numbers is a universal class. The universal class will be denoted by the symbol **V** in conformity with the notation of Peano. The class $-\mathbf{V}$ is then the null class and will be denoted by the symbol $\mathbf{\Lambda}$. In arithmetic an example of the null class is the *class consisting of the roots of the equation:* $x + 1 = x$.

The late Stanisław Leśniewski did not like the concept of the *null class*. He cited the following passage from Frege in support of his position :

> " If a class is composed of objects, is a set, a collective combination of objects, it must disappear if these objects disappear. By burning all the trees of a forest, we burn the forest itself. Consequently there cannot be a null class." [4]

Leśniewski boasted that :

> " . . . throughout my life there was, on the whole, no time when I was not in agreement with this concise remark." [5]

One could tell that throughout Leśniewski's life there was, on the whole, no time when he understood the concept of a class. A forest is not a class, and a class is not a collective

[1] Padoa : *l.c.*, p. 57.
[2] Padoa : *l.c.*, p. 57.
[3] George Boole : *The Laws of Thought*, reprinted in the *Collected Logical Works* as vol. ii, Chicago, 1916, pp. 46–52.
[4] Gottlob Frege : " Kritische Beleuchtung einiger Punkte in E. Schröder's 'Vorlesungen über die Algebra der Logik' ", *Archiv für systematische Philosophie*, Bd. 1 (1895), pp. 436–7, cited by Stanisław Leśniewski, " O podstawach matematyki," (" On the Foundations of Mathematics,") *Przegląd filozoficzny*, tom. 30, 1927, p. 196.
[5] Leśniewski : *l.c.*

combination of objects, nor need it disappear if these objects disappear.

If the concept:

$$\textit{the root of the equation}: \; x + 1 = x$$

is considered, it is clear that it does not denote numbers and therefore does not denote the roots of the equation: $x + 1 = x$. This equation has no roots. The concept is a null class.

If null classes were rejected as illegitimate constructions, a theory of equations could not be developed. The question might be raised whether the function $f(x)$ becomes zero at some point. To discover the answer to this question, the equation: $f(x) = 0$ must be investigated. This cannot be done if null classes are held to be illegitimate constructions. Similarly the so-called *reductio ad absurdum* proof could not be employed in geometry.

A critique of the assumptions upon which the development of science has been based is absolutely necessary and is one of the main ways of obtaining a view of the world which is based upon sound reason. However, a critique which is based upon common misunderstandings and which is imposed on the reader by the use of powerful dialectical devices, must lead to confusion of concepts.

Leśniewski spoke as if it were a common mistake to employ null classes. Actually mathematicians are to be censured only because they interpret classes as material collections. Consequently it is indeed difficult to understand what a null class is. But just as it does not follow that there are no infinitely small numbers, merely because the differentials of Newton and Leibniz were not defined precisely and intelligibly, it does not follow that there are no null classes.

Logic and mathematics can deal only with certain expressions. Obviously therefore classes are certain expressions. Null classes are equally good expressions as universal classes. Nothing further need be said.

Leśniewski's argument that in daily life null classes are not handled cannot be sustained. It can be confirmed repeatedly that certain classes which are regarded as universal classes are in fact null classes. Progress could not be made if discussion were limited to non-null classes; discussion of unknown stars, unknown roots, etc., would be precluded. Artificial language like that of Leśniewski would have to be employed.

Peirce, MacColl, Jevons, Venn, and Schröder continued the work of Boole and De Morgan.

In 1867 Peirce discovered the distributive property of the sign $\cup$ with respect to the sign $\cap$. In the symbolism of this book the theorem in question may be written:

$$\underset{Cl}{=} \cup C \cap A B \cap \cup C A \cup C B$$ [1]

In 1878 MacColl discovered the following laws:

(1) $\equiv \subset A \cap B C \wedge \subset A B \subset A C.$

This is the distributive law of the sign $\subset$ with respect to the sign $\cap$.[2]

(2) $\equiv \subset \cup B C A \wedge \subset B A \subset C A$

Huntington brought these investigations to a definitive conclusion.[3] He formulated the calculus of classes with the help of a table of axioms, which is given below in a slightly modified form, i.e. the construction rules are separated from the general axioms.

1. Construction rules.

(*a*) If A is a class, then $- A$ is a class.
(*b*) If A and B are classes, then $\cup A B$ is a class.
(*c*) If A and B are classes, then $\cap A B$ is a class.

2. Axioms.

I*a* $\underset{Cl}{=} \cup A \mathbf{\Lambda} A$

I*b* $\underset{Cl}{=} \cap A \mathbf{V} A$

II*a* $\underset{Cl}{=} \cup A B \cup B A$

II*b* $\underset{Cl}{=} \cap A B \cap B A$

III*a* $\underset{Cl}{=} \cap \cup A B \cup A C \cup A \cap B C$

III*b* $\underset{Cl}{=} \cup \cap A B \cap A C \cap A \cup B C$

IV*a* $\underset{Cl}{=} \cap A - A \mathbf{\Lambda}$

IV*b* $\underset{Cl}{=} \cap A - A \mathbf{V}$

V $\sim \underset{Cl}{=} \mathbf{\Lambda} \mathbf{V}$ [4]

[1] Cf. Padoa: *l.c.*, p. 58.
[2] Cf. Padoa: *l.c.*, p. 59.
[3] Cf. Edward V. Huntington: "Sets of Independent Postulates for the Algebra of Logic," *Transactions of the American Mathematical Society*, July, 1904, vol. 5, pp 292–3.
[4] Cf. A. N. Whitehead and B. Russell: *Principia Mathematica*, vol. 1, Cambridge, 1910, pp. 205–6.

2. The failure to determine precisely the *univers du discours*, i.e. to specify the objects which may be discussed, was the essential defect of the calculus of classes. Paradoxes resulted from this obscurity when Cantor passed from the calculus of classes to the general theory.[1]

The concepts of elementary semantics and the calculus of propositions permit the calculus of classes to be placed upon firm foundations.

All the propositions of elementary semantics which can be obtained from the pattern $\mathbf{\Pi}\, I\, F$, i.e. all those which begin with the general quantifier, will be regarded as classes. In particular the following are classes: $\mathbf{\Pi}\, \bar{a} = \bar{a}\, \bar{a}$, $\mathbf{\Pi}\, \bar{a} = \bar{a}\, \mathbf{0}$, $\mathbf{\Pi}\, \bar{a} = \bar{a}\, \bar{a}$, $\mathbf{\Pi}\, \bar{a}\, \{ * * \mathbf{0\,0\,0}\, \bar{a} \}$, $\mathbf{\Pi}\, \bar{a}\, (* \bar{a}\, \bar{a} * \mathbf{0\,0\,0}\, \bar{a})$.

The first of these classes is the universal class, since $= E\, E$ *is* always *true*. $\mathbf{0}$ is the only element of the class $\mathbf{\Pi}\, \bar{a} = \bar{a}\, \mathbf{0}$, since it alone satisfies the condition $= E\, \mathbf{0}$ *is true*. The class $\mathbf{\Pi}\, \bar{a} \sim = \bar{a}\, \bar{a}$ is a null class, because no expression satisfies the condition $\sim = E\, E$ *is true*. The class $\mathbf{\Pi}\, \bar{a}\, \{ * * \mathbf{0\,0\,0}\, \bar{a} \}$ has three elements, i.e. the expressions $\mathbf{0}$, $* \mathbf{0\,0}$, $* * \mathbf{0\,0\,0}$ because these expressions alone satisfy the condition $\{ * * \mathbf{0\,0\,0}\, E \}$ *is true*. The class $\mathbf{\Pi}\, \bar{a}\, (* \bar{a}\, \bar{a} * \mathbf{0\,0\,0}\, \bar{a})$ is the class composed of the infinitely many expressions which satisfy the condition $(* E\, E * \mathbf{0\,0\,0}\, E)$ *is true*. It is these expressions which are called *whole numbers*.

If J is an element of the class F, this fact is expressed by the pattern: $\varepsilon\, J\, F$. The following theorems can be proved:

$$\varepsilon\, u\, \mathbf{\Pi}\, \bar{a} = \bar{a}\, \bar{a},$$

$$\varepsilon\, \mathbf{0}\, \mathbf{\Pi}\, \bar{a} = \bar{a}\, \mathbf{0},$$

$$\sim \varepsilon\, u\, \mathbf{\Pi}\, \bar{a} = \bar{a}\, \bar{a},$$

$$\wedge\, \varepsilon\, \mathbf{0}\, \mathbf{\Pi}\, \bar{a}\, \{ * * \mathbf{0\,0\,0}\, \bar{a} \}\, \wedge\, \varepsilon * \mathbf{0\,0}\, \mathbf{\Pi}\, \bar{a}\, \{ * * \mathbf{0\,0\,0}\, \bar{a} \}$$
$$\varepsilon * * \mathbf{0\,0\,0}\, \mathbf{\Pi}\, \bar{a}\, \{ * * \mathbf{0\,0\,0}\, \bar{a} \},$$

$$\supset \varepsilon\, u\, \mathbf{\Pi}\, \bar{a}\, (* \bar{a}\, \bar{a} * \mathbf{0\,0\,0}\, \bar{a})\, \varepsilon * u\, u\, \mathbf{\Pi}\, \bar{a}\, (* \bar{a}\, \bar{a} * \mathbf{0\,0\,0}\, \bar{a}).$$

If the pattern $\varepsilon\, A\, B$ is employed the calculus of classes can be reduced to the calculus of propositions. To perform this reduction it is necessary to accept the following additional rule, which will be called the *transformation rule:*

If in the propositional function F the apparent variable K, not contained in F, is substituted for the real variable I, and if

[1] Cf. Georg Cantor: *Contributions to the Founding of the Theory of Transfinite Numbers*, translated by Philip E. B. Jourdain, Chicago, 1915.

the result of this substitution is G, then $\equiv F\,\varepsilon\, I\, \Pi\, K\, G$ *is a theorem.*[1]

For example the following are theorems:

$$\equiv (* 0\, u * 0\,0\,0\, u)\, \varepsilon\, u\, \Pi\, \bar{a} (* 0\, \bar{a} * 0\,0\,0\, \bar{a})$$
$$\equiv \{ * 0 * 0\,0\, u \}\, \varepsilon\, u\, \Pi\, \bar{a} \{ * 0 * 0\,0\, \bar{a} \}, \text{ etc.}$$

If the pattern $\phi\,(\,X\,)$ is employed

$$\equiv \phi\,(\,u\,)\, \varepsilon\, u\, \Pi\, \bar{a}\, \phi\,(\,\bar{a}\,)$$

In addition to this rule only the following abbreviations are necessary:

Abbreviation	Expression
$\dot{-}\ A$	$\Pi\, \bar{a} \sim \varepsilon\, \bar{a}\, A$
$\frown A\ B$	$\Pi\, \bar{a} \wedge \varepsilon\, \bar{a}\, A\, \varepsilon\, \bar{a}\, B$
$\smile A\ B$	$\Pi\, \bar{a} \vee \varepsilon\, \bar{a}\, A\, \varepsilon\, \bar{a}\, B$
$= A\ B$ Cl	$\Pi\, \bar{\bar{x}} \equiv \varepsilon\, \bar{\bar{x}}\, A\, \varepsilon\, \bar{\bar{x}}\, B$
$\subset A\ B$	$\Pi\, \bar{\bar{x}} \supset \varepsilon\, \bar{\bar{x}}\, A\, \varepsilon\, \bar{\bar{x}}\, B$
$\mathbf{V}$	$\Pi\, \bar{a} = \bar{a}\, \bar{a}$
$\mathbf{\Lambda}$	$\Pi\, \bar{a} \sim = \bar{a}\, \bar{a}$

The reduction cannot be regarded as complete because the symbol $\varepsilon\, A\, B$ and the transformation rule have been accepted. Rational metamathematics, which will be introduced later, permits a complete reduction to be obtained.

To become familiar with the role of the above abbreviations, the following examples will be considered.

The following are theorems:

$$\underset{\text{Cl}}{=} \frown \Pi\, \bar{a}\, (*0\bar{a}*000\bar{a}) \Pi\, \bar{a} \{**000\bar{a}\} \Pi\, \bar{a}\, \Lambda\, (*0\bar{a}*000\bar{a}) \{**000\bar{a}\}$$

$$\underset{\text{Cl}}{=} \frown \Pi\, \bar{a} \{ * * 0\,0\,0\, \bar{a} \} \Pi\, \bar{a} \{ 0\, \bar{a} \} \Pi\, \bar{a} = \bar{a}\, 0$$

$$\underset{\text{Cl}}{=} \smile \Pi\, \bar{a} = \bar{a} 0 \smile \Pi\, \bar{a} = \bar{a} * 00 \Pi\, \bar{a} = \bar{a} * * 000 \Pi\, \bar{a} \{ * * 0\,0\,0\, \bar{a} \}$$

It will now be shown that

$$\subset \Lambda\, A,$$

is a theorem, where A is any class.

Let the class $\Pi\, \bar{a} \sim = \bar{a}\, \bar{a}$ and the class pattern $\Pi\, \bar{a}\, \phi\,(\,\bar{a}\,)$ be substituted for Λ and A respectively. In conformity with the calculus of propositions

$$\supset \sim = u\, u\, \phi\,(\,u\,)$$

since a false proposition implies any proposition.

[1] The necessity for such a rule was pointed out to me by the late Stanisław Leśniewski. I first introduced it in *T. C. T.*, p. 25.

If the transformation rule is applied, the theorem:

$$\supset \varepsilon u \Pi \bar{a} \sim = \bar{a}\bar{a} \varepsilon u \Pi \bar{a} \phi (\bar{a})$$

will be obtained.

By the introduction of the symbol $\boldsymbol{\Lambda}$ for $\Pi \bar{a} \sim = \bar{a}\bar{a}$ and the letter A for the pattern $\Pi \bar{a} \phi (\bar{a})$,

$$\supset \varepsilon u \boldsymbol{\Lambda} \varepsilon u A$$

is obtained.

The application of the rule of generalization yields the theorem:

$$\Pi \bar{x} \supset \varepsilon \bar{x} \boldsymbol{\Lambda} \varepsilon \bar{x} A$$

which on abbreviation becomes the desired theorem.

It should be observed that the methods given above do not permit the proof of the general theorem. However, it has been pointed out that it can be proved for each separate class.

The following example will now be considered:

The expression $\Pi \bar{a} \{ * \mathbf{0} * \mathbf{0}\,\mathbf{0}\, \bar{a} \}$ can be regarded as the class of expressions contained in $* \mathbf{0} * \mathbf{0}\,\mathbf{0}$. On the other hand the expression

$$\Pi \bar{a}\ \mathsf{V} = \bar{a}\mathbf{0}\ \mathsf{V} = \bar{a} * \mathbf{0}\,\mathbf{0} = \bar{a} * \mathbf{0} * \mathbf{0}\,\mathbf{0}$$

represents the class, whose elements are the expressions: $\mathbf{0}$, $* \mathbf{0}\,\mathbf{0}$, $* \mathbf{0} * \mathbf{0}\,\mathbf{0}$. It can be shown that both classes contain the same elements, i.e. that the two classes are equal.

First the theorem:

$$\equiv \{ * \mathbf{0} * \mathbf{0}\,\mathbf{0}\, u \}\ \mathsf{V} = u \mathbf{0}\ \mathsf{V} = u * \mathbf{0}\,\mathbf{0} = u * \mathbf{0} * \mathbf{0}\,\mathbf{0}$$

will be proved.

The proof of this theorem is based upon the analysis of the expression $* \mathbf{0} * \mathbf{0}\,\mathbf{0}$ into its component expressions in accordance with the pattern $= * E\, F\, G$.

If the transformation rule is applied to this theorem, the theorem

$$\equiv \varepsilon u \Pi \bar{a} \{ * \mathbf{0} * \mathbf{0}\,\mathbf{0}\, \bar{a} \} \varepsilon u \Pi \bar{a}\ \mathsf{V} = \bar{a}\mathbf{0}\ \mathsf{V} = \bar{a} * \mathbf{0}\,\mathbf{0} = \bar{a} * \mathbf{0} * \mathbf{0}\,\mathbf{0}$$

will be obtained.

The application of the rule of generalization to this theorem yields:

$$\Pi \bar{x} \equiv \varepsilon \bar{x} \Pi \bar{a} \{ * \mathbf{0} * \mathbf{0}\,\mathbf{0}\, \bar{a} \} \varepsilon \bar{x} \Pi \bar{a}\ \mathsf{V} = \bar{a}\mathbf{0}\ \mathsf{V} = \bar{a} * \mathbf{0}\,\mathbf{0} = \bar{a} * \mathbf{0} * \mathbf{0}\,\mathbf{0}$$

This theorem can be written in the following form:

$$= \Pi \bar{a} \{ * \mathbf{0} * \mathbf{0}\,\mathbf{0}\, \bar{a} \} \Pi \bar{a}\ \mathsf{V} = \bar{a}\mathbf{0}\ \mathsf{V} = \bar{a} * \mathbf{0}\,\mathbf{0} = \bar{a} * \mathbf{0} * \mathbf{0}\,\mathbf{0}$$

Cl

This theorem can be read as follows: The class $\Pi \bar{a} \{ * \mathbf{0} * \mathbf{0}\,\mathbf{0}\, \bar{a} \}$ contains the same elements as the class $\Pi \bar{a}\ \mathsf{V} = \bar{a}\mathbf{0}\ \mathsf{V} = \bar{a} * \mathbf{0}\,\mathbf{0} = \bar{a} * \mathbf{0} * \mathbf{0}\,\mathbf{0}$.

The calculus of relations was developed parallel to the calculus of classes. Until recently it was regarded as an independent discipline. However, not long ago Kuratowski showed that it can be reduced to the calculus of classes.[1]

Instead of talking about the relation R it is possible to speak about the class of pairs of elements, between which this relation holds. For example, instead of talking about the relation which holds between father and son, it is possible to speak about the class of pairs of men in which the first element is the father and the second element the son.

In other words the concept of an ordered pair of elements to the concept of a class.

Kuratowski's procedure was as follows: From two elements X and Y, where X occupies the first place and Y the second, he forms the classes E and F such that E contains only one element X, and F is the class of elements X and Y. The class whose elements are the classes E and F represents the pair of elements X and Y. It is obvious that the element X belongs to both classes E and F, while the element Y belongs only to the class F. If X is identical with Y, the classes E and F reduce to one class, and the pair of elements (X, X) which completely characterizes the element X, is employed.

The calculus of relations can be reduced to the calculus of propositional functions in a way analogous to that which was employed in the case of the calculus of classes.

The relation between two expressions, where the first expression contains the second, will be represented by the symbol:

$$\mathbf{\Pi}\,\bar{x}\,\mathbf{\Pi}\,\bar{y}\,\{\,\bar{x}\,\bar{y}\,\}$$

The converse relation will be represented by the symbol:

$$\mathbf{\Pi}\,\bar{x}\,\mathbf{\Pi}\,\bar{y}\,\{\,\bar{y}\,\bar{x}\,\}$$

The proposition: $\mathbf{rel}\,\mathbf{\Pi}\,\bar{x}\,\mathbf{\Pi}\,\bar{y}\,H\,(\,I\;K\,)$ *is true* will be read:

I stands to K in the relation $\mathbf{\Pi}\,\bar{x}\,\mathbf{\Pi}\,\bar{y}\,H$

The transformation rule is as follows:

If in the propositional function F, the apparent variables K and H are substituted for the real variables I and E respectively, where neither H nor K is contained in F, and if the result of this substitution is G, then $\equiv F\ \mathbf{rel}\,\mathbf{\Pi}\,\bar{a}\,\mathbf{\Pi}\,\bar{b}\,G\,(\,J\;H\,)$ *is a theorem.*

[1] Cf. Casimir Kuratowski: "Sur la notion de l'ordre dans la théorie des ensembles," *Fundamenta Mathematicae*, tom. 2, Warszawa, 1921, pp. 161–171. See also N. Wiener, "A Simplification of the Logic of Relations," *Proceedings of the Cambridge Philosophical Society*, vol. 17, 1912–14, pp. 387–390.

In particular

$$\equiv \{ u\ v \} \mathbf{rel}\, \mathbf{\Pi}\, \bar{a}\, \mathbf{\Pi}\, \bar{b} \{ \bar{a}\, \bar{b} \} (* 0 0 0) \{ * 0 0 0 \}$$

$$\equiv \{ u\ v \} \mathbf{rel}\, \mathbf{\Pi}\, \bar{a}\, \mathbf{\Pi}\, \bar{b} \{ \bar{a}\, \bar{b} \} (0 * 0 0) \sim \{ 0 * 0 0 \}$$

are theorems.

Once again a table of abbreviations is posited. This table is as follows :

Abbreviation	Expression
$.\,\dot{=}\, A$	$\mathbf{\Pi}\, \bar{a}\, \mathbf{\Pi}\, \bar{b} \sim \mathbf{rel}\, A (\bar{a}\, \bar{b})$
$.\frown A\ B$	$\mathbf{\Pi}\, \bar{a}\, \mathbf{\Pi}\, \bar{b} \wedge \mathbf{rel}\, A (\bar{a}\, \bar{b})\, \mathbf{rel}\, B (\bar{a}\, \bar{b})$
$.\smile A\ B$	$\mathbf{\Pi}\, \bar{a}\, \mathbf{\Pi}\, \bar{b} \vee \mathbf{rel}\, A (\bar{a}\, \bar{b})\, \mathbf{rel}\, B (\bar{a}\, \bar{b})$
$=_{\mathbf{Rel}} A\ B$	$\mathbf{\Pi}\, \bar{x}\, \mathbf{\Pi}\, \bar{y} \equiv \mathbf{rel}\, A (\bar{x}\bar{y})\, \mathbf{rel}\, B (\bar{x}\bar{y})$
$.\,\mathbf{(}\ A\ B$	$\mathbf{\Pi}\, \bar{x}\, \mathbf{\Pi}\, \bar{y}\ \mathbf{)}\ \mathbf{rel}\, A (\bar{x}\bar{y})\, \mathbf{rel}\, B (\bar{x}\bar{y})$
$\mathbf{V}_{\mathbf{Rel}}$	$\mathbf{\Pi}\, \bar{a}\, \mathbf{\Pi}\, \bar{b} \wedge = \bar{a}\, \bar{a} = \bar{b}\, \bar{b}$
$\mathbf{\Lambda}_{\mathbf{Rel}}$	$\mathbf{\Pi}\, \bar{a}\, \mathbf{\Pi}\, \bar{b} \wedge \sim = \bar{a}\, \bar{a} \sim = \bar{b}\, \bar{b}$

If X and Y satisfy the condition $\mathbf{rel}\, R (X\ Y)$ *is true*, X is called the *antecedent* and Y the *consequent* of the relation R. The class of the antecedents and consequents of the relation R is called the *field* of this relation.

3. To account for the fruitfulness of the calculus of classes it suffices to see the way in which it permits the elimination of the concept of segment from the arithmetic of real numbers. In accordance with Dedekind's[1] method, arbitrary classes of rationals, which are less than a certain whole number will be regarded as real numbers.

The following relations will be postulated between real numbers conceived in this way and rational numbers :

(*a*) The null class is *arithmetically equal to 0.*

(*b*) The class containing elements greater than every rational number which is less than the rational number E and not containing elements greater than E, is arithmetically equal to E.

(*c*) The class all of whose elements are less than a certain rational number which is less than the rational number E, is *arithmetically less than E.*

(*d*) The class which is neither arithmetically less than nor arithmetically equal to E and which is a real number, is *arithmetically greater than E.*

For example, the class of rationals less than $\frac{1}{2}$, the class

[1] Cf. Richard Dedekind · *Essays on the Theory of Numbers*, translated by Wooster Woodruff Beman, Chicago, 1901, " Continuity and Irrational Numbers," pp. 15–19.

whose only element is $\frac{1}{2}$, and the class whose elements are obtained from the pattern $\frac{1}{n}$, where $n = 2, 3, \ldots$ are seen to be real numbers which are arithmetically equal to $\frac{1}{2}$.

Real numbers are compared by applying the following rules:

The number E is *greater* than the number F, if one of the elements of the number E is greater than the number F.

The number E is *equal to* the number F, if neither E is greater than F, nor F is greater than E.

Operations are performed on the real numbers E and F by performing them on pairs of elements of these numbers.

For example, $(E + F)$ is the class of sums obtained by adding an arbitrary element of the class E to an arbitrary element of the class F.

Obviously the systematic construction of the arithmetic of real numbers requires considerable work, but the fundamental conception is very simple.

It should also be noted that $\sqrt{2}$ is simply the class of rationals whose square is less than 2.

Such real numbers as π, the base of the natural logarithms, e, etc., can be constructed in similar fashion.

If it is observed that the semantical method permits classes to be regarded as certain expressions, it will be easy to understand that expressions are the concrete objects which replace pseudo-concrete segments in arithmetic. Thus arithmetic ceases to be a science about ideal objects.

It is clear that the construction of the universal class of real numbers is impossible. If it is constructed Richard's paradox will result. However, just as it was possible to construct classes of segments, it is possible to construct separate classes of real numbers and to talk about the *upper limit* of such classes.

Thus this method permits a simple definition of the upper limit of a class of real numbers.

The upper limit of a class of real numbers is simply the class of rationals which are elements of at least one of the real numbers of the given class.

If this class of rationals contains elements which are greater than any whole number, the upper limit of the given class is said to be infinite. Otherwise it is a real number.

If a given interval of real numbers (α_0, β_0) is divided successively into parts by introducing the intervals of numbers

(α_1, β_1), (α_2, β_2), (α_3, β_3) in such a way that $\alpha_n < \alpha_{n+1}$, $\alpha_{n+1} < \beta_{n+1}$, $\beta_{n+1} < \beta_n$, where $n = 0, 1, 2, \ldots$, the existence of the upper limit of the numbers α_n is obviously a trivial matter. The upper limit of the numbers α_n is the class of rationals which are elements of at least one of the numbers α_n. It is a real number because its elements are all less than any rational number and are arithmetically greater than β_0.

This method simplifies the proofs of theorems in elementary geometry concerning the proportionality of segments, the measurement of areas and circumferences, etc.

4. The classical theory of classes developed from an extremely idealistic, platonic view of the world. It is the only case in the history of modern science of a fruitful reaction to platonic idealism.

Georg Cantor was the creater of the theory of classes. His only precursor was the Austrian clergyman, Bernard Bolzano (1781–1848). Bolzano believed in the actual infinite. He devoted his entire life to showing that the paradoxes attached to the infinite are illusory.[1] He invented the idea of a one to one correspondence between two infinite classes. The class of even numbers can be made to correspond in a one to one manner to the class of whole numbers by the equation :

$$k = 2 \cdot i \,.$$

It is clear that to any whole number i corresponds the even number $2 \cdot i$. Conversely to any even number k corresponds the whole number $(k : 2)$. It follows that the class of even numbers is similar to the class of whole numbers since the former is part of the latter. Bolzano maintains that the paradox which might be seen in this fact is only apparent, because the essential characteristic of the infinite is that part of an infinite class can be similar to that class.

Cantor extended the investigations of Bolzano on the similarity of classes. In particular he proved that the class of rationals is similar to the class of whole numbers.[2] Classes which are similar to the class of whole numbers are called denumerable. In other words it may be said that denumerable classes have the property that from their elements a sequence can be formed. It may be seen that from the elements of the class of rationals the following sequence may be formed :

If only irreducible expressions are considered, i.e. fractions

[1] Federigo Enriques : *The Historic Development of Logic*, authorized translation by Jerome Rosenthal, New York, 1929, p. 137.
[2] Cf. Georg Cantor : *l.c.*, pp. 43–4, p. 111.

which cannot be simplified, the fractions, the sum of whose numerator and denominator is 1 are placed first. Since there is but one such fraction: $\frac{0}{1}$, the first expression of the sequence is therefore 0. Irreducible fractions the sum of whose numerator and denominator is 2, are considered next. Since there is only one such fraction, $\frac{1}{1}$, the second term of the sequence is therefore 1. There are two irreducible fractions the sum of whose numerator and denominator is 3. They are: $\frac{1}{2}$, and $\frac{2}{1}$. The smaller is placed first. The third term of the sequence is therefore $\frac{1}{2}$ and the fourth is 2. In this way the following sequence is obtained:

$$0,\ 1,\ \frac{1}{2},\ 2,\ \frac{1}{3},\ 3,\ \frac{1}{4},\ \frac{2}{3},\ \frac{3}{2},\ 4,\ \frac{1}{5},\ 5,\ \frac{1}{6},\ \frac{2}{5},\ \frac{3}{4},\ \frac{4}{3},\ \frac{5}{2},\ 6\ \ldots$$

It is easy to give the general construction rule for this sequence. If the sums of the numerator and denominator of two irreducible fractions are computed, the fraction with the smaller sum precedes the fraction with the larger sum. If the sums of the numerator and denominator of the two fractions are equal that fraction which is smaller precedes.

Cantor's principle discovery was based upon the proof that the real numbers do not form a denumerable class,[1] i.e. that there is no denumerable class of real numbers which contains all the real numbers.[2] Cantor did not hesitate to operate with the concept of the class of all real numbers. In consequence this concept is generally employed in contemporary mathematics without any questions being raised in connection with it. Mathematicians disregarded the objections of Richard and Poincaré because they held that semantical questions are not part of mathematics.

Cantor's second great discovery was the conception of well-ordered classes.[3]

[1] Cf. Cantor: *l.c.*, p. 38 f.

[2] The proof of this theorem was given in 3.8 and the difficulties which arise in connection with the conception of the *class of all real numbers* were also pointed out.

[3] Cantor: *l.c.*, pp. 60–1, 75 ff., 137 ff.

A class is said to be well-ordered, if each of its not-null subclasses has a first element.

The class of rationals is obviously not well-ordered with regard to magnitude, since, e.g., the class of rationals greater than **1** has no first element.

However, every sequence is a well-ordered class.

The terms of a sequence of sequences can be ordered in such a way that the term a_{ik} precedes the term $a_{i(k+m)}$ and the term $a_{(i+m)n}$. In other words, of two terms which have the same first index and therefore belong to the same sequence, the term which is earlier in the given sequence is regarded as earlier. Of two terms which belong to two different sequences the term which belongs to the earlier of the two sequences is regarded as earlier.

If, for example, the sequence of sequences:

$$\begin{array}{l} a_{1\,1},\ a_{1\,2},\ a_{1\,3},\ a_{1\,4},\ \ldots \\ a_{2\,1},\ a_{2\,2},\ a_{2\,3},\ a_{2\,4},\ \ldots \\ a_{3\,1},\ a_{3\,2},\ a_{3\,3},\ a_{3\,4},\ \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

is considered, it is easy to show that its terms can be well-ordered on the basis of the above rule.

It will be agreed now that two ordered classes are to be called ordinally similar if a one to one correspondence can be set up between them in such a way that the image of an earlier element is invariably earlier than the image of a later element.[1]

Classes of well-ordered ordinally similar classes may be regarded as numbers.[2]

First there are finite ordinal numbers, which correspond to whole numbers. They are succeeded by the ordinal number ω, which is the class of denumerable classes. ω is succeeded by the number $\omega + 1$, which is the class of classes such that:

$$1, 2, 3, \ldots, \omega,$$

i.e. the class which is obtained by adding still another term to the terms of the sequence.

It is clear that the possibilities of forming well-ordered classes are unlimited.

The reader will feel that these ideas are very alluring. "Eodem modo literis atque arte animos delectari posse." [3]

[1] Cantor: *l.c.*, p. 112.

[2] Whitehead and Russell: *Principia Mathematica*, Cambridge, 1912, vol. 2, p. 291 ff.

[3] Cantor's motto, cited by Adolf Fraenkel: *Einleitung in die Mengenlehre*, Berlin, 1928, 3 Aufl., p. 1, n. 2.

Unfortunately difficulties similar to those which were encountered in the theory of real numbers are met at every step.

The theory of ordinals enabled Cantor to construct the theory of cardinals. By the cardinal number of a class is understood the class of classes similar to a given class.[1]

The class of classes similar to the class of natural numbers is denoted by $\aleph_0$.

The class of classes similar to the class of the ordinal numbers of denumerable classes is denoted by the symbol $\aleph_1$.

It can be shown that classes whose cardinal number is $\aleph_1$, like the class of real numbers, are not denumerable. The problem whether $\aleph_1$ is the cardinal number of the class of real numbers has not been solved. This problem is conventionally decided in the affirmative by accepting the so-called *hypothesis of the continuum.*

5. The fundamental difficulties encountered in these magnificent conceptions led to the great drama in Cantor's life and gave evidence of his genius.

Professor Adolf Fraenkel writes [2]:

> "Rarely in the history of mathematics and perhaps in the history of all science has such a vast discipline been created and developed to such heights by a single man, as the theory of aggregates was developed by its creator, Georg Cantor (1845–1918). In the face of the opposition of almost the entire mathematical world he pursued and maintained his ideas."

In the introduction to his excellent text on the theory of aggregates, Fraenkel writes as follows [3]:

> "Beside the fruitful intuition and the creative power of a genius, which guided Cantor in his discoveries, unusual energy and persistence on his part were necessary to apply his intuitions and carry his point regarding them; for a long time, even up to the last decade of the nineteenth century when Cantor had already concluded his literary activity (1897), much to his sorrow, his intuitions were attacked by the vast majority of his mathematical contemporaries (above all by L. Kronecker) as obscure and false, or at the very least—by his well-wishers—as 'having been propounded a century too early'." . . .

[1] Whitehead and Russell: *l.c.*, p. 13.

[2] A. Fraenkel: "Die Entstehung der Mengenlehre," *Scientia*, vol. 48, 1930, p. 361.

[3] A. Fraenkel: *Einleitung in die Mengenlehre*, Berlin, 1928, 3 Aufl., pp. 1–2.

Cantor's spirit was broken by his opponents. It could hardly be foreseen that in time, even those of his successors who rejected his metaphysics but sought to defend those of his results which have decided the fate of mathematics, would be combatted with equal fervour.

Whether disputes about new views concerning the foundations of a science will always result in misery, contempt, and erroneous opinions can hardly be foreseen. In any event such has always been the case up to the present day. Thinkers who have developed familiar ideas have always been honoured and secured material well-being. Conquerors of new domains of thought have always been exposed to and threatened with danger.

Cantor's theory is an apparent triumph of idealism. At all events it manifests his creative power.

If all idealists were of the type of Husserl and his pupils, the struggle with idealism might be pointless. But a consideration of Cantor's genius shows that it was inseparably connected with a firm belief in the world of ideal objects and an almost mystical religiosity. Consequently all desire for a thorough-going critique of the bases of human genius is lost. We are possessed by the same perpetual desire to glorify our creations to eternity as were the Pythagoreans.

Human nature is strangely weak and capricious. It is stronger than the experience accumulated during the course of centuries. However, it must be understood once and for all that the value of all thoughts which have ever been produced depends upon some mysterious profundity, and their metaphysical character, if any, does not affect their scientific value. What Cantor directly associated with the actual infinite was his personal survival, which has nothing in common with the fruitfulness of his ideas. The greatness of his doctrines has nothing in common with the mysterious allure of the infinite. It lies in the fact that certain operations can be performed with the help of signs which have been defined. However, it must be kept in mind that if Cantor had confined himself to performing these operations and if the fires of a mystical yearning for infinity had not burned in his soul, he would never have discovered the simple laws which he left mankind as a permanent acquisition (*aere perennius*).

However, it must be confessed that Cantor's metaphysics became the source of tragic errors which decided its fate.

He employed the term *aggregate*.

> "By an 'aggregate' (Menge) we are to understand any collection into a whole (Zusammenhang zu einem Ganzem) M of definite and separate objects m of our intuition or our thought. These objects are called the 'elements' of M." [1]

The reader can readily observe that this concept has the characteristics of such contradictory constructions as Ingarden's heteronomical expressions which were described above.[2] The discovery of this fact was made by Bertrand Russell.

In constructing Russell's famous antinomy, it should be kept in mind that according to Cantor, every class (aggregate) is a definite, separate object of our thought and can therefore in turn be regarded as an element of a class. In particular it can be asked whether or not a given class is an element of itself. For example, the class of non-men is an element of itself because this class is not a man. Similarly the class of all classes must be an element of itself, etc. The class of all classes is an object so sublime, that it might arouse a desire to seek the exact relationship between the theory of classes and theology. In view of Russell's construction, dreams of this kind disappear.

Russell formed the *class of all classes which are not elements of themselves.*

In our symbolism the expression

$$\Pi \bar{x} \sim \varepsilon \bar{x} \bar{x}$$

represents this class.

This expression will be denoted by the letter Ω and the proposition

$$\varepsilon \Omega \Omega$$

will be considered.

The application of the principle of transformation to this proposition yields the equivalence:

$$\equiv \varepsilon \Omega \Omega \sim \varepsilon \Omega \Omega$$

Thus the proposition $\varepsilon \Omega \Omega$ is equivalent to its own negation.

In ordinary language this paradox can be expressed as follows:

If Ω is the class of classes which are not elements of themselves, Ω can be neither an element of itself nor not an element of itself.

If Ω is an element of itself, it is not a class which is not an element of itself and therefore cannot be an element of itself.

[1] Cantor: *l. c.*, p. 85. [2] Cf. 2. 4.

If Ω is not an element of itself, it is a class which is not an element of itself and therefore is an element of itself.

It follows immediately from this antinomy that the question whether or not a class is an element of itself has no precise and determinate meaning.

Russell's discovery was one of the most dramatic awakenings ever experienced by science. The entire structure of the theory of classes which had required such effort to erect and which had promised so much, ceased to be a clear and simple edifice. A flaw appeared in its foundations which threatened complete catastrophe.

Gottlob Frege wrote at the end of the second volume of his *Grundgesetze der Arithmetik*:

> "Hardly anything more unwelcome can happen to a scientific writer, than that after the completion of a work, one of the foundations of his edifice should be shaken. I was placed in this position by a letter from Mr. Bertrand Russell, when the printing of this volume was nearly completed."

And a little later:

> "*Solatium miseris socios habuisse malorum.* If it is any consolation, all those who have employed the extension of concepts, classes, and aggregates in their proofs are in a situation similar to mine." [1]

The Italian mathematician, Burali-Forti,[2] was the author of the first published antinomy. This antinomy can be formulated as follows:—

If the class of all ordinals is considered, it forms a well-ordered class and therefore can be employed in creating a new ordinal, which must be greater than all the ordinals.

This paradox is perhaps even more disturbing than that of Russell because of its uncanny simplicity.

Both these paradoxes irrefutably show that Cantor's naïve concept of a class cannot be maintained.

However, it does not follow that Cantor's theory is invalidated. It will be seen that it can be saved by placing very simple restrictions upon the language to be employed and that it can be reconstructed in such a way that the elements of metaphysical idealism which are involved in it, will be preserved.

[1] Gottlob Frege: "Nachwort," *Grundgesetze der Arithmetik*, Bd. 2, Jena, 1903, p. 253

[2] Cesare Burali-Forti: "Una questione sui numeri transfiniti," *Rendiconti del Circolo Matematico di Palermo*, vol. 11, 1897, pp. 154–164.

But the dream of an absolute mathematics which is independent of logical constructions vanished.

6. To remove Russell's antinomy it suffices to accept the simplified theory of logical types. Bertrand Russell was the creator of the theory of types. This theory is complicated and cannot be clearly formulated in a few words. However, it can be simplified in such a way that it can be explained by a few simple sentences.

A so-called *univers du discours* which is composed of objects called individuals will be accepted. No further properties of these individuals nor any concrete examples of them will be given.

Classes of individuals, classes of classes of individuals, etc., will also be accepted.

Thus clearly the concept of a class as such has no meaning. Only classes composed of certain determinate objects can be discussed. Consequently the question whether or not a class is an element of itself is meaningless.

I formulated the simplified theory of types for the first time in an article published in 1921, which contains the following sentence [1]:

> "To remove this antinomy (Russell's antinomy is concerned here) the simple theory of types, which depends upon the distinction between individuals, functions of individuals, functions of these functions, etc., is sufficient."

In this context the concept of a function may be regarded as equivalent to the concept of a class.

The following year I formulated the same conception.[2]

In 1925 I published an article based upon the simplified theory of types.[3] Nevertheless I have never maintained that this theory definitively settles the foundation-problems of logic.

In 1925 F. P. Ramsey advanced such a thesis and referred to my article of 1922, although from a completely different point of view.[4] Ramsey's article rapidly gained popularity. Professor

[1] *A. L. F.*, cited by Alonzo Church: "A Bibliography of Symbolic Logic," *The Journal of Symbolic Logic*, vol. 1, 1936, p. 172, no. 220 (3).
[2] *U. A. P. M.*, p. 241.
[3] *U. H. M.*, p. 439 ff.
[4] F. P. Ramsey: "The Foundations of Mathematics," *Proceedings of the London Mathematical Society*, Ser. 2, vol. 25, 1926, reprinted in *The Foundations of Mathematics and Other Logical Essays*, London, 1931, pp. 1–61.

Carnap in a recently published article unreservedly recognized my priority in regard to this matter.[1]

It should be added that in 1925 Professor Wilkosz constructed a theory of classes, based upon the simplified theory of types.[2]

The simplified theory of types is very important because it permits simple intuitive foundations to be obtained for the theory of classes and does not require the use of a symbolic apparatus.

Yet this theory does not permit everyday language to be employed to its fullest extent. In particular semantical considerations must be abandoned if the antinomy of Richard is to be avoided.

The method of constructing the concept of real number on the basis of the simplified theory of types will not be given here. It is a fact that if we accept the so-called axiom of infinity from which it follows that the class of classes of individuals is infinite, it is possible to construct the whole theory of real numbers without difficulty as was proved by Whitehead and Russell.[3] Real numbers can be conceived as certain classes of rationals. In virtue of the simplified theory of types all classes of rationals are of the same type. It therefore follows that the Cantorian number on the diagonal is of the same type as the numbers which form the fundamental sequence. The consequence of this state of affairs is the antinomy of Richard. This antinomy can be avoided only if the theory of classes is acknowledged to be a closed discipline from which all semantical investigations are excluded.

This is a very odd state of affairs. On the one hand we speak of all the properties of the real numbers. On the other hand we neglect the properties of real numbers which can be represented by the expressions of our system. It is as if we were concerned only with those properties which actually can be constructed in our system. Thus a careful consideration of the implications of this conception seems to show that the idealism involved in the simplified theory of types is illusory. If the class of real numbers is said to be non-denumerable and if this assertion does not lead to contradiction in spite of the fact that we are actually concerned with the numbers constructible with our system, such a statement can be made

[1] Cf. Rudolf Carnap: "Die Antinomien und die Unvollständigkeit der Mathematik," *Monatshefte für Mathematik und Physik*, Bd. 41, 1934, p. 265.

[2] Wilkosz: *Podstawy ogólnej teorji mnogości* (*Foundations of a General Theory of Aggregates*), Kraków, 1925.

[3] Whitehead and Russell: *l.c.*, pp. 183.

only because our system provides no means to form the sequence of these numbers.[1]

From this point of view the actual infinite disappears and what is called *non-denumerability* is but an awkward description of the poverty of our system.

Thus the theory of types is a strange mixture of idealistic and nominalistic elements.

The idealistic character of this doctrine manifests itself in the supposition that there exist infinitely many classes of individuals about which nothing is established.

Furthermore if we do not wish to accept the altogether superfluous and completely metaphysical axiom of extensionality, it is possible to prove the existence of a non-constructive class in our system.[2]

The determining factor leading to the rejection of the simplified theory of types is its incompatibility with the axioms of semantics. Detailed investigations on the foundations of the exact sciences are impossible if semantical constructions are not studied in full detail. From this point of view the simplified theory of types has outlived its value. It must be replaced by a new and much more general logic. The simplified theory of types is of course well-suited for semi-intuitive investigations. But such investigations can be carried on just as easily with the help of the axiom system given by Professor Zermelo, which was perfected by Professor Fraenkel. It is also possible, following Hausdorff and Sierpiński to disregard the difficulties involved in the theory of classes, to avoid complicated constructions, and to be governed by sound intuition. It is well known that this method has proved very fruitful and led to a great increase in the number of investigations on the theory of classes. A large number of such investigations have been made on Polish soil. The journal, *Fundamenta Mathematicae,* founded by the late Mr. Janiszewski and published in Warsaw by Professors Sierpiński, Mazurkiewicz, and Kuratowski contains a mass of results which are difficult to understand but which are of great value for investigators working on the foundations of mathematics. The further development of these investigations was hindered by grave difficulties connected with the fact that a concept of a class which is not associated with the concept

[1] Carnap gave this interesting interpretation of the theory of types in Carnap: *l.c.*

[2] Cf. *N. G. M.*, p. 369 f.

of an expression is not uniquely defined. These difficulties clearly bring out the fact that in time the theory of classes will become part of that much more general science, rational metamathematics.

7. These difficulties will be considered in 6.8. I would now like to say a few words about the *pure theory of logical types*. Bertrand Russell developed this theory in such a way that the use of intuition was required. According to this theory neither classes of individuals, nor classes of classes of individuals, etc., form a definite logical type; they fall rather into types which in this case can be called orders.

To avoid misunderstandings it should be observed that Russell speaks of *propositional functions* and not of classes. By a class he understands a certain fictional construction which cannot be given in simple fashion. I will not go into these matters here, but confine myself to the remark that this method proved to be unfortunate and led to great confusion of concepts. I have already fully criticized Russell's conception of classes.[1]

I will rather consider his theory of types.

Predicative classes of individuals are the lowest type of classes. They cannot be defined; neither is it possible to construct an example of such a class within the system of logic. It is therefore clear that this view is extremely idealistic in character. If now any class of predicative classes of individuals ω is formed, its sum and product can be constructed. The sum of the classes ω, which is denoted by $\Sigma\,\omega$, is the class of individuals belonging to a certain predicative class which is an element of the classes ω.

The product of the classes ω, which is denoted by $\Pi\,\omega$, is the class of individuals belonging to all classes of individuals which are elements of the classes ω.

The classes $\Sigma\,\omega$, $\Pi\,\omega$ are examples of non-predicative classes of individuals. They are constructions of a higher order. These constructions form a separate type of objects.

If now we have objects of any type it is possible to construct predicative classes of these objects. The sums and products of these classes provide objects of a new type, etc.

This theory has been called the *branched* theory of types, because, as is clear, the types do not form a sequence.

In order to reconstruct the theory of Cantor with the help of this radical theory, Russell added the following idea. He

[1] Cf. *T. C. T.*, *T. C. T.*, II.

accepted an axiom which he called the *axiom of reducibility*. This axiom says that *to every class corresponds a certain predicative class which is equivalent to it.* In view of this axiom, the classes $\Sigma\,\omega$, $\Pi\,\omega$ have their equivalents among the predicative classes of individuals. In practice, therefore, the situation is such that in speaking of all predicative classes of individuals we also have in mind those classes which contain the same elements as $\Sigma\,\omega$, or $\Pi\,\omega$, and propositions concerning constructive classes are never considered. In the entire theory of classes predicative equivalents of given classes are employed. These equivalents are obviously non-constructive. We are satisfied with supposing that they exist.

This is clearly an idealistic view in which great advances have been made over that of Cantor. This theory can be said to be the highest triumph of idealism because it is clear that it eliminates all fear of the occurrence of paradoxes in the theory of classes and also of the occurrence of semantical paradoxes, such as that of Richard. Idealism cannot be combatted on the grounds of contradiction alone. If it were possible to derive a contradiction from the idealistic views, this matter would be settled once and for all. However, the problem of existence is not so simple. The belief in the existence of ideal objects can never be completely overthrown. We can only contrast concrete constructions with it and show that everything which is fruitful and productive in idealism can be obtained with the help of these constructions.

It is possible to obtain thereby everything but a certain state of mind which is, for certain people, synonymous with the beauty of life. Exact arguments cannot counteract this factor. I can only make the following comments.

I myself have experienced both the state of mind which accompanies idealistic metaphysics and the pleasures which result from it. However, it is accompanied by serious danger. All is well so long as the rigours of life do not actually touch us. Just as the turtle locks himself up in his shell, we shut ourselves up and look disdainfully at the wickedness about us, until a brutal hammer crushes our shell, leaving only sorrow and solitude. It would be much better to combat the difficulties of idealism at an initial stage. This struggle affords far more pleasure than does an idealistic state of mind.

Russell's method of removing Richard's paradox will now be considered. Every real number can be regarded as a class of pairs composed of a whole number which indicates a certain decimal place, and one of the numbers **0**, **1**, **2**, . . .,**9** which indicates the number found at that place.

I assert that the class of predicative classes of this type is non-denumerable. Actually the Cantorian number is not a predicative class, but it follows from the axiom of reducibility that there exists a predicative equivalent of this number. This equivalent cannot belong to the sequence which was employed in constructing the Cantorian number. This number was actually constructed, but its equivalent which is not constructible is what is concerned. Consequently all attempts to express it in semantical terms are to be scorned.

The axiom of reducibility is a typical synthetic apriori proposition. On the appearance of Russell's theory, Poincaré pointed out this fact.[1] If logic were based upon such axioms, it could not be regarded as a science which is independent of metaphysics. If no other path were open, metaphysics would be the chief science and Plato would have to be regarded as the discoverer of the mystery of existence. Fortunately such is not the case. It was shown that the pure theory of types could be developed independently of the concept of predicative classes. The investigations which I conducted along these lines led ultimately to the erection of foundations for logic which involve no metaphysical presuppositions.

In particular it was shown that the rejection of the principle of reducibility did not involve the overthrow of Whitehead and Russell's fundamental idea of constructing a system of the apriori sciences which would be based upon logic. It is necessary to accept only the axiom of infinity, which, however, cannot be regarded in the same way as the axiom of reducibility. The latter principle postulates the existence of non-constructive objects. In the case of the former we appeal to the existence of any number of different individuals and therefore to something dealt with directly in semantics. From this point of view, the axiom of infinity combines both logical and semantical elements. A system of logic based upon the pure theory of types and the axiom of infinity permits the reconstruction of classical mathematics and fails to include only the theory of Cantor.[2] From the point of view of philosophy this result is very important. The exact sciences can then

[1] Cf. Henri Poincaré : " La logique de l'infini," *Revue de métaphysique et de morale*, vol. 17, 1909, pp. 461–482. Reprinted in *Dernières Pensées*, Paris, 1913, as Chapter IV, pp. 101–139.

[2] Cf. *T. C. T.*, ii, pp. 92, 137.

be made independent of metaphysics and only certain portions of them are connected with metaphysics. Nevertheless this state of affairs seems to show that in a certain sense idealism is necessary and that there are heights of human thought which it is impossible to attain by nominalistic methods. This conclusion is undoubtedly false and requires severe criticism. The supposition must be made that this situation may be like that observed in geometry. The view that geometry is an idealistic science obviously is derived from the fact that it is not an independent science but merely a certain method of interpreting the theory of numerical functions. May not logic likewise be a dependent science? Do not perhaps its weaknesses arise from the fact that it has been artificially separated from semantics, which necessarily is the basis of all sciences? The reader who has studied the previous chapters with care will not hesitate to answer this question in the affirmative. Logic without semantics is a fragment and it is not strange that the use of the methods of logic does not take us very far. It will be seen later that not only nominalistic logic but even the extremely idealistic logic of Russell must be supplemented by new hypotheses as mathematics develops more and more. Therefore it is impossible to hesitate. It is necessary to set about constructing a system of rational metamathematics based upon the axioms of semantics. It will be seen that this method liquidates the claims of idealism once and for all by attaining its most mysterious heights.

The semantical method enables us to formulate the original conception of logical types of Richard and Poincaré.

It suffices to say that expressions used in speaking about given expressions must be of a higher type than those expressions. In this way is obtained a sequence of expressions, each member of which is of a higher type than the previous member. Such a hierarchy of expressions will be constructed below.

A theory of expressions based upon this rule can be built up in many ways. I have worked out such a theory in collaboration with Messrs. Hetper and Herzberg.[1]

Another much more natural theory will be introduced in full detail in Chapter VII.

8. The difficulties which have been mentioned in 6, 6, in connection with the construction of a theory of classes will now be considered.

[1] Cf. *F. M. R.*

I am concerned above all with the famous *axiom of choice* or *selection*, also called the *principle of Zermelo*.[1]

It is clear that to every not null class of natural or rational numbers there belongs a *selected element* which can be regarded as its representative.

In the first case the selected element is a natural number which is not greater than any element of the given class.

In the second case the selected element is a rational number which is not later than any rational number in the sequence of rationals given above.

If now any class of real numbers is imagined. If the class is given *explicitly* and if it can be proved of at least one of the real numbers, that it is an element of that class, this number which was chosen freely can be said to be the representative of the class. But such an individual choice is not very useful because it can be made only in a limited number of cases. Moreover it is impossible to dream of a construction rule which would enable us to assign a representative to every class of real numbers as was done in the case of classes of natural numbers. This state of affairs greatly restricts the theory of classes and reduces it to a useless fragment. To get out of this difficult situation, Zermelo proposed the acceptance of the following axiom: *to every class of classes belongs a class of representatives of this class.*

Since Zermelo's axiom is independent of the other axioms of the theory of classes, only[2] the following problems are of interest here:

(1) Is it possible to construct a fruitful theory of classes in which Zermelo's principle would be false?

(2) Is it possible to construct a theory of classes in which Zermelo's principle would be unnecessary?

The second question can be answered in the affirmative since the theory of classes can be regarded as a part of rational metamathematics. In rational metamathematics only denumerable classes are treated. They can therefore be representatives of classes. The only question which remains concerns the domain of the metamathematical theory of classes, since we would not like to have to abandon the

[1] Cf. Ernst Zermelo: "Beweis, dass jede Menge wohlgeordnet werden kann," *Mathematischen Annalen*, Bd. 59, 1904, pp. 514–16. "Untersuchungen über die Grundlagen der Mengenlehre," i, *ib.* Bd. 65, 1908, p. 266.

[2] Cf Adolf Fraenkel: "Untersuchungen über die Grundlagen der Mengenlehre," *Mathematische Zeitschrift*, Bd. 22, 1925, pp. 250–273.

important results obtained by means of the theory of classes in recent decades. There are significant indications that it may be unnecessary to do so. A detailed analysis of this matter will be made in the following chapter.

The first of the two questions formulated above will now be considered. It can also be decided in the affirmative. My investigations on the role of Zermelo's principle in the theory of classes,[1] led to the conclusion that there is a striking analogy between this principle and the postulate of Euclid. This analogy indicates that it is possible to construct a theory of classes as fruitful as that of Cantor in which Zermelo's principle is false.

The simplified theory of types and the axiom of infinity will be taken as the basis of our inquiry. In a system based upon these suppositions it is possible to prove the existence either of classes of individuals which contain *n* elements where *n* is any whole number, or of classes to which all individuals belong with the exception of *n* individuals, where again *n* is any whole number.

While it is true that I did not succeed in proving this theorem, it is obvious to every one who is familiar with the methods of investigations of the theory of classes. But even the reader comparatively unfamiliar with this subject will understand that so long as it is supposed with regard to individuals only that infinitely many classes of classes of individuals can be constructed we will not have means which are sufficient to construct sequences of individuals. In such a case the only means of constructing classes of individuals which is at our disposal is to write down a certain number of letters and to suppose that these letters denote different individuals.

This state of affairs is described by the following principle which I have called the *principle of transcendence:*

If a *denotes a certain class of individuals, either* a *or* $-a$ *must contain n individuals when n is a whole number.*

It follows from this principle that an infinite sequence of individuals cannot exist. Otherwise the class of the even elements of this sequence would be infinite, as would the class of all other individuals. This state of affairs contradicts the principle of transcendence.

If an infinite sequence of individuals cannot exist, neither can a well-ordered class of individuals exist. But it follows from Zermelo's principle that to every class of individuals

[1] Cf. *U. H. M.*

belongs a relation which establishes the well-ordering of its elements. This is the famous theorem of Zermelo, published in 1904 in an article which Fraenkel regards as amazingly acute.[1] Thus it necessarily follows from this theorem and from a consequence of this principle of transcendence that there exists no sequence of individuals, that the principle of choice is false.

On the basis of the axiom of transcendence I constructed a theory of whole generalized numbers[2] to which complete induction is not applicable. This result indicates that in a certain sense the axiom of transcendence extends our domain of investigation.

To prevent the loss of these theorems of a theory of classes, which is based upon Zermelo's principle, it is sufficient to accept the following axiom:

The class of classes whose elements are well-ordered is likewise well-ordered.

It follows from this axiom that the real numbers are well-ordered.

This axiom is obviously but one of the many axioms of this kind which might have been accepted. It is therefore clear that only roughly defined concepts are employed here.

It should be observed that Zermelo's principle is not sufficient to define the concept of class. It is not adequate for solving the previously mentioned problem regarding the hypothesis of the continuum.

Furthermore it is impossible to predict whether the decision of this hypothesis in the affirmative would remove the indefiniteness involved in the concept of class.

All this indicates that the theory of classes is not an independent study and is a powerful argument for the necessity of employing semantical methods in investigations concerning aggregates.

[1] Cf. Fraenkel: *Einleitung in die Mengenlehre, l.c.*, p. 196.
[2] Cf. *U. H. M.*, p. 462.

CHAPTER VII

FOUNDATIONS OF RATIONAL METAMATHEMATICS

1. Intuitive metamathematics was developed by Hilbert and his school. Hilbert regarded metamathematics as a device to be employed in proving the consistency of the axioms of idealistic mathematics; he did not, however, attempt to formulate the methods of metamathematics precisely. Because of the development of elementary semantics the attempt to do so can be made successfully.

Hilbert's view is deceptive. It is now known that when metamathematics is correctly constructed, the introduction of special mathematical axioms is unnecessary. It is therefore possible to eradicate completely all idealistic elements from mathematics without imposing any restrictions upon the domain of metamathematical investigations.

If the so-called epsilon axiom:

$$\supset A\,(a)\,A\,(\varepsilon A)\ [1]$$

is considered, the idealistic character of Hilbert's axiom system becomes very evident.

This apparently innocent little formula conceals within itself a whole metaphysics. It says that in every class A which is not a null class, a distinct element εA is contained. The only justification for this axiom lies in the fact that it makes the calculi simpler.

If, for example, the class **K** of irrational numbers is considered, it is clear that none of these numbers is better or worse than any other. None of the elements of class **K** is distinct. But if the existence of such an element is assumed and the symbol ε**K** is introduced in a definition, the illusion that an actual object is involved is created. This is a typical *deus ex machina*. I recall that I hit upon a similar idea when I was a student at Göttingen. I wrote to Bertrand Russell about it and he replied that by such a procedure I would not gain anything and would only disguise the existing difficulty.

So much effort was expended by Hilbert and Bernays [2]

[1] David Hilbert: "Die Grundlagen der Mathematik," Gekürzter Abdruck aus den *Abhandlungen des mathematischen Seminars zu Hamburg*, Bd. 6, 1928, pp. 65–85, reprinted as "Anhang IX", *Die Grundlagen der Geometrie*, 7 Aufl., Leipzig, 1930, p. 292.

[2] Cf. D. Hilbert and P. Bernays: *Grundlagen der Mathematik*, Bd. II, Berlin, 1939.

in defending this axiom, and so many complications ensue, that the practical advantage resulting from its acceptance is almost nil. Hermann Weyl's opinion on this subject is very similar to mine, but he only goes so far as to doubt its advantage.[1]

This objection would be decisive even if Hilbert had succeeded in proving the consistency of his axioms. But as a matter of fact no such proof has been given and there is no indication that such a proof will ever be given. Up to the present time the work of von Neumann and others on this subject seems to indicate that proofs of consistency can be given only if the system in question involves constructible objects alone.[2]

Hilbert's procedure conceals grave methodological dangers; it justifies anything which is at all useful and which does not lead to a contradiction, even though it may be sheer nonsense. As is known, magic, astrology, and the like cannot be shown to be absurd and it is probable that their utility can be successfully defended. For example, the supposition that there exist higher beings, who can exert no influence upon us, never leads to a contradiction, and is of great use to the supporters of the hierarchy of classes. Nevertheless the fact remains that this supposition is a fiction which contradicts sound reason and greatly hampers the development of intellectual culture.

2. There are undoubtedly portions of mathematics which are very useful to us, even though they are based upon constructions which as yet are not entirely clear. Such is not the case with regard to metamathematics. Metamathematics yields nothing fundamentally new. Its problem is merely the attainment of a higher degree of certainty, the elimination of metaphysical presuppositions and the reduction of proofs to elementary forms. If such a science is to have any value it must be formulated precisely. That which is presupposed and the procedure to be followed must be explicitly stated. This requirement seems to involve a vicious circle for if that upon which mathematics depends is specified, the problem seems only to have been pushed one step further back. That upon which metamathematics depends would now have to be specified and consequently a metamathematics of higher order

[1] Cf. H. Weyl: "Diskussionsbemerkungen zu dem zweiten Hilbertschen Vortrag über die Grundlagen der Mathematik," *Abhandlungen des mathematischen Seminars zu Hamburg, l.c.*, p. 87.

[2] Cf. J. v. Neumann: "Zur Hilbertschen Beweistheorie," *Mathematische Zeitschrift*, Bd. 26, 1927, pp. 1–46.

would be required. This is by no means the case. It will be seen that it is possible to begin with a very simple finitistic system which will be called the auxiliary system. The system of elementary semantics can be constructed with the help of this system. With the aid of both these systems an infinite sequence of systems of metamathematics can then be constructed such that each subsequent system can be investigated in the preceding system.

The construction of the fundamental system of semantics will now be begun. First the system will be given without any explanations. Actually only the rules of the system are necessary to construct the expressions and theorems of the system. However, a purely mechanical construction of expressions and theorems would not be very economical. It is necessary to interpret the constructions of the system to obtain the particular expressions and theorems in which we are interested. Such an interpretation is also necessary in order to become convinced that the system really describes the operations performed in intuitive semantics. Consequently the rules are analysed and a few examples are given.

The system which is formulated here has as yet not been published. It is a modification of the system published in collaboration with Hetper.[1] The changes introduced were motivated by the desire for the greatest possible simplicity.

The unwieldiness of expressions and apparent tediousness which results from following the rules of this system are an unavoidable consequence of the fact that everyday language is confined to a minimum. The abbreviations introduced in the various tables reduce long expressions to a form which is easier to handle. Actually the system is conceived in such a way that it might have been developed without the use of abbreviations. The construction of such a system would be facilitated if there were a machine at our disposal to designate very long starred expressions and automatically analyse them into simpler expressions. It must always be kept in mind that actually the directions given here are for the use of such a machine.

I think that those readers who do not wish to learn the symbolism given here confirm the fact that the least possible use has been made of everyday languages in constructing this system.

In contrast to the systems of Whitehead and Russell, the

[1] Cf. *N. F. F. M.*

system of Hilbert and Bernays and my own system of the pure theory of types, everyday language is confined to a minimum in this system.

The language employed may be represented by the following patterns :

a. **E** *is an expression,*
b. **E** *is a theorem,*
c. If **X**, then **Y**,

where **E** denotes any expression, and **X** and **Y** denote any propositions of the form *a*, *b*, or *c*. The rules of intuitive reasoning which are employed are the rule of substitution and the rule of modus ponens. Obviously the unconscious use of negative or universal propositions is excluded. It may be seen that the intuitive logic which is employed is much poorer than that of Brouwer. From the point of view of the logic which will be presented, negation is an operation of higher order which first appears when a symbolic language is employed. This state of affairs is compatible with the theory of Steinberg who holds that direct affirmation of facts never has a negative form. His theory is based upon that of Sigwart,[1] who regarded negative propositions as propositions about propositions. I think that this is worthy of note because on the basis of this view it is possible to understand the fact that the productivity of intuitive thought has its origin in the construction of concrete objects.

I should also like to point out that the superiority of a method based upon the use of a restricted everyday language over other methods, was stressed very emphatically by A. F. Bentley and Max Black.

3. The rules of the fundamental system of semantics consist of

(1) The rules for expressions,
(2) The rules for the auxiliary system,
(3) The rules for the proper systems,

which will be denoted by $(R\ E)$, $(R\ A)$, and $(R\ P)$ respectively. These rules will now be built up with the help of abbreviations. It should be noted that while abbreviations permit diffuse expressions to be avoided they are not proper elements of the fundamental system of semantics. No rules for the use of abbreviations will be assumed. Only when these abbreviations

[1] Christoph von Sigwart : *Logic*, translated by Helen Dendy, New York, 1895, vol. 1, p. 119.

are eliminated will correct theorems of the system be obtained. It is obvious that this could be done by means of a special machine.

Neither descriptions nor names will be employed in building up expressions. The expressions are given simply as material objects. In our system any occurrence of an expression can be replaced by any other occurrence of it, i.e. in the language of this system no distinction is to be made between the various occurrences of a given expression.

The letters **E**, **F**, **G**, **H**, **J**, **K**, **L**, **M**, **N**, **X**, **Y**, **Z** are employed as real variables in the fundamental system of semantics. Their domain of substitution is defined by the rules (*R E*). The use of these variables might be avoided but complications of the rules would result.[1]

To construct the theorems of the fundamental system, the following rules are employed:

1. substitution of expressions for real variables,
2. modus ponens.

(*R E*)

1. **c** *is an expression.*
2. If **E** *and* **F** *are expressions,* then ⁎ **E F** *is an expression.*

By means of these rules it is possible to construct as many expressions as are desired. Since in accordance with rule 1 the letter **c** is an expression, it can be taken as a value of the variables in rule 2. After the verification of the hypothesis of 2 in this way the application of the rule of modus ponens yields:

⁎ **c c** *is an expression.*

Similarly

⁎ **c** ⁎ **c c** *is an expression.*
⁎ ⁎ **c c c** *is an expression.*
⁎ ⁎ **c c** ⁎ **c c** *is an expression,* etc.

The following abbreviations will be employed to construct the theorems of auxiliary system.

If **E**, **F**, **G**, **H** and **L** are expressions

.0 L **.1 L** **.2 L** . . .	is an abbreviation of	⁎ **L L** **.0 .0 L** **.0 .1 L** . . .	Integers of type **L.**

[1] Cf. *K. P. Z.*, pp. 290–1.

.I L .II L	is an abbreviation of	* .0 L .1 L * .1 L .0 L	Auxiliary asymmetrical expressions.
(E F G H) [L]		* * * * .I L .0 E .0 F .0 G .0 H	The fundamental pattern of substitution.
/ E F		* .I E .II F	The fundamental pattern of Sheffer's stroke.
~ F ⊃ E F ∨ E F ∧ E F ≡ E F		/ F F / E / F F / / E E / F F / / E F / E F / / / E E / F F / E F	Pattern of functions of the logical calculus.
= E F [L]		(E .0 L .0 L F) [L]	Pattern of identity.
{E F} [L]		∨ = F .0 L [L] ~ (E F .0 L E) [L]	Pattern of inclusion.
Prop E		⊃ E E	Pattern of propositions.
Expr [L] E		= E E [L]	Pattern of expressions of type **L**.
0 1 2 . . . $\overline{10}$ $\overline{11}$ . . .		.0 c .1 c .2 c . . . .$\overline{10}$ c .$\overline{11}$ c . . .	(Integers of type **c**) Fundamental integers.
I II		.I c .II c	Fundamental auxiliary asymmetrical expressions.
(E F G H) = E F {E F} Expr E		(E F G H) [c] = E F [c] {E F} [c] Expr [c] E	Semantical patterns of type **c**.

HETPER'S RULES FOR THE AUXILIARY SYSTEM[1] (*RA*)

1. $\mathbf{Expr\ 0}$ *is a theorem.*

If $\mathbf{Expr\ E}$, $\mathbf{Expr\ F}$, $\mathbf{Expr\ G}$, $\mathbf{Expr\ H}$, $\mathbf{Expr\ J}$ and $\mathbf{Expr\ K}$ *are theorems,* then

2. $\mathbf{Expr} * \mathbf{E\,F}$ *is a theorem.*

3.1 $\sim = \mathbf{0} * \mathbf{E\,F}$ *is a theorem.*

3.2 $\sim = * \mathbf{E\,F\,0}$ *is a theorem.*

3.3 If $\sim = \mathbf{E\,F}$ *is a theorem,* then $\sim = * \mathbf{E\,G} * \mathbf{F\,H}$ *is a theorem.*

3.4 If $\sim = \mathbf{E\,F}$ *is a theorem,* then $\sim = * \mathbf{G\,E} * \mathbf{H\,F}$ *is a theorem.*

4.1 $(\mathbf{0} * \mathbf{E\,F\,G\,0})$ *is a theorem.*

4.2 $(\mathbf{E\,E\,F\,F})$ *is a theorem.*

4.3 If $\sim = \mathbf{G} * \mathbf{E\,F}$, $(\mathbf{E\,G\,H\,J})$, and $(\mathbf{F\,G\,H\,K})$ *are theorems,* then $(* \mathbf{E\,F\,G\,H} * \mathbf{J\,K})$ *is a theorem.*

5. If $\sim = \mathbf{H\,J}$ and $(\mathbf{E\,F\,G\,H})$ *are theorems,* then $\sim (\mathbf{E\,F\,G\,J})$ *is a theorem.*

6.1 If $\mathbf{E}$ and $\mathbf{F}$ *are theorems,* then $\sim / \mathbf{E\,F}$ *is a theorem.*

6.2 If $\mathbf{E}$ and $\sim \mathbf{F}$ *are theorems,* then $/ \mathbf{E\,F}$ *is a theorem.*

6.3 If $\sim \mathbf{E}$ and $\mathbf{F}$ *are theorems,* then $/ \mathbf{E\,F}$ *is a theorem.*

6.4 If $\sim \mathbf{E}$ and $\sim \mathbf{F}$ *are theorems,* then $/ \mathbf{E\,F}$ *is a theorem.*

It can easily be seen that these rules conform to the rules of the elementary semantical calculus and the elementary logical calculus.

EXAMPLES

(1) $\sim = \mathbf{0\,1}$ *is a theorem* [3.1]

(2) $(\mathbf{0\,1\,0\,0})$ *is a theorem* [4.1]

(3) $(\mathbf{1\,1\,0\,0})$ *is a theorem* [4.2]

(4) $\sim = \mathbf{1} * \mathbf{0\,1}$ *is a theorem* [(1), 3.4]

(5) $(* \mathbf{1\,1\,0\,1})$ *is a theorem* [(4), (2), (3), 4.3]

(6) $\sim (* \mathbf{0\,1\,1\,0} * \mathbf{0\,1})$ *is a theorem* [(4), (5), 5]

(7) $\sim / \sim (* \mathbf{0\,1\,1\,0} * \mathbf{0\,1}) \sim (* \mathbf{1\,1\,0} * \mathbf{0\,1})$ *is a theorem* [(6), (6), 6.1]

(8) $\sim \sim \sim (* \mathbf{0\,1\,1\,0} * \mathbf{0\,1})$ *is a theorem* [(7)]

(9) $/ \sim = \mathbf{0\,1} \sim \sim (* \mathbf{0\,1\,1\,0} * \mathbf{0\,1})$ *is a theorem* [(1), (8), 6.2]

(10) $\mathbf{V} = \mathbf{0\,1} \sim (* \mathbf{0\,1\,1\,0} * \mathbf{0\,1})$ *is a theorem* [(9)]

(11) $\{* \mathbf{0\,1\,1}\}$ [(10)]

[1] Cf. Władysław Hetper: "Zagadnienie zupełności elementarnej semantyki," "The Problem of the Completeness of Elementary Semantics," *Archiwum Towarzystwa Naukowego we Lwowie* (unpublished).

It should be noted that Hetper's auxiliary system leads to significant simplifications of semantical methods. It can be proved without any difficulty that it is consistent and decidable.

INTERPRETATION OF THE FUNDAMENTAL PATTERNS OF THE AUXILIARY SYSTEM

If **E F G** and **H** *are expressions,* if **Prop (E F G H)**, **Prop Y** and **Prop Z** *are theorems,* then

H is the result of the substitution of G for F in E.	is the interpretation of	**(E F G H) is a true symbolic proposition.**
Either it is not the case that Y is a true symbolic proposition or it is not the case that Z is a true symbolic proposition.		**/ Y Z is a true symbolic proposition.**

Application

Prop (0 0 0 0) *is a theorem.*
Prop ((0 0 0 0) 0 0 (0 0 0 0)) *is a theorem.*
Prop / (0 0 0 0) ((0 0 0 0) 0 0 (0 0 0 0)) *is a theorem.*

0 is the result of the substitution of 0 for 0 in 0.	is the interpretation of	**(0 0 0 0) is a true symbolic proposition.**
(0 0 0 0) is the result of the substitution of 0 for 0 in (0 0 0 0).		**((0 0 0 0) 0 0 (0 0 0 0) is a true symbolic proposition.**
Either it is not the case that 0 is the result of the substitution of 0 for 0 in 0, or it is not the case that (0 0 0 0) is the result of the substitution of 0 for 0 in (0 0 0 0).		**/ (0 0 0 0) ((0 0 0 0) 0 0 (0 0 0 0)) is a true symbolic proposition.**

In the symbolic proposition **/ (0000) ((0000)00(0000))** the first occurrence of **(0 0 0 0)** is called a *propositional component,* the other occurrences of **(0 0 0 0)** are called *semantical components.* It is obvious that the method of interpretation which has been given permits the elimination of the propositional components of a given proposition. Only its semantical components are present in the final result. It is also clear that this method of interpretation is unique and is compatible with ordinary language. However, since the expressions have no intrinsic meaning but only an assigned meaning, there can be no doubt concerning their use.

The patterns of expressions **(EFGH)[L]**, and **/EF** have been employed to build up the propositions of the auxiliary system. The expression **L** is the *type* of the pattern of substitution.

The following table contains expressions of the lowest types :

Type	Expressions	Original expressions
c	**0, *00, *0*00, **000, **00*00**	**0, *0*00, **000**
0	**1, *11, *1*11, **111, **11*11**	**1, *1*11, **111**
1	**2, *22, *2*22, **222, **22*22**	**2, *2*22, **222**

It is clear that the types of expressions are built up from **c** and the fundamental integers.

Expressions which can be built up from the star *****, and the fundamental integer **.0K**, are expressions of type **K**.

Any expression of type **K** which is not an expression of type **.0K** is an original expression of type **K**.

The patterns of the expressions to be found in the abbreviations are invariant with respect to type. For example, **(EFGH)[L]** can denote either **(EFGH)[c]**, **(EFGH)[0]** or **(EFGH)[1]**, etc.

In this way patterns of substitution of types **c**, **0**, **1**, etc., are obtained.

The expressions are built up in conformity with Hetper's theory of mutually independent patterns. This method leads to the elimination of all ambiguity of interpretation, for all values of the real variables.[1]

4. The following table of abbreviations will now be accepted:

If **E**, **F**, **G**, **H**, **J**, **K**, **L**, **M**, **N**, **X**, **Y**, **Z** are expressions, then

[1] Cf. W. Hetper: "Rola schematów niezależnych w budowie systemu semantyki," "The Role of Independent Patterns in the Structure of the System of Semantics," *Archiwum Towarzystwa Naukowego we Lwowie*, Dział III, tom IX, 1938, pp. 253–264.

0_L 1_L . . . 10_L . . .	is an abbreviation of	.0 L .0 0_L . . . .0 9_L . . .	Integers of type **L**.
Integ [L] E **Integ E**		**(.0 E 1_L 0_L E) [L]** **Integ [c] E**	The propositional functions of integers.
$\alpha_{K\ L}$ $\beta_{K\ L}$ . . .		* * .I K 1_K .I L * * .I K 2_K .I L . . .	Propositional variables.
$\bar{\alpha}_L$ . . .		$\alpha_{.0\ L\ L}$ . . .	
$\bar{\bar{\alpha}}_L$ . . .		$\alpha_{.1\ L\ L}$ . . .	
$a_{K\ L}$ $b_{K\ L}$ . . .		* * .Π K 1_K .I L * * .Π K 2_K .I L . . .	Semantical variables.
$\bar{a}_L$ . . .		$a_{.0\ L\ L}$ . . .	
$\bar{\bar{a}}_L$ . . .		$a_{.1\ L\ L}$ . . .	
Π [M N] X E Π [M N] X Y E . . .		* * * .I M .0 X .0 E .I N Π [M N] X Π [M N] Y E . . .	General quantifiers.

The symbols α and β occurring in tables on pp. 171, 180, 185, 311–312, and in notes on pp. 308, 309, 316, have the same significance as those (α and β) occurring in text on pp. 173, 175, 176, 177, 186, 306, 307, 308, 318.

∃ [M N] X E ∃ [M N] X Y E . . .	is an abbreviation of	∼ Π [M N] X ∼ E ∃ [M N] X ∃ [M N] Y E . . .	Particular quantifiers.
Ax log E F G H		⊃ / E / F G ⊃ / H F / E H	Nicod's syllogism.
Ax sem$_1$ E F G H [L]		⊃ ∨ = E .0 L [L] = H .0 L [L] ≡ (E F G H) [L] ∨ ∧ = H E [L] ∼ = E F [L] ∧ = F E [L] = H G [L]	The semantical axiom of start.
Ax sem$_2$ E F G H J K [L]		≡ (⋆ E F G H ⋆ J K) [L] ∨ ∧ = H ⋆ J K [L] = G ⋆ E F [L] ∧ ∼ = G ⋆ E F [L] ∧ (E G H J) [L] (F G H K) [L]	The semantical axiom of recursion.
Ax sem$_3$ E F [L]		∼ = .0 L ⋆ E F [L]	Semantical axiom of diversity
Ax sem E F G H J K [L]		∧ Ax sem$_1$ E F G H [L] ∧ Ax sem$_2$ E F G H J K [L] Ax sem$_3$ E F [L]	The semantical axiom.
⊨ [M N] E		⋆ ⋆ .I M .0 E .Π N	The pattern of theorems of the system [M N]
((E F G H)) [L] ((E F G H))		∧ (E F G H) [L] ∧ {E F} [L] ∼ {E G} [L] ((E F G H)) [c]	The special substitution.
gen [K L]$_H$ (E J) F G gen [K L] (E J) F G		∧ Π [M N] ⋆ ⋆ J E .I H (F ⋆ ⋆ J E .I H ⋆ ⋆ J E .I L G) [H] ∧ {F ⋆ ⋆ J E .I H} [H] ∼ {F ⋆ ⋆ J E .I L} [H] gen [K L]$_c$ (E J) F G	The pattern of generalization.

The expressions in this table are the patterns of the concepts of the system **[M N]**, or symbolic descriptions of these concepts. They will be elucidated after the rules of proper systems have been formulated.

The rules for the construction of proper systems will now be stated.

(*R P*)

If **E**, **F**, **G**, **H**, **J**, **K**, **L**, **M**, **N**, **X**, **Y**, **Z** are expressions, then

1. Axioms

If $\wedge\ \mathbf{Integ\,M} \wedge \{\mathbf{M\,L}\} \wedge \{\mathbf{L\,.0\,N}\}\,\mathbf{Expr\,N}$ *is a theorem,* then

1.1 $\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ \mathbf{.0\,L}$ *is a theorem.*
[Axiom of constant expressions.]

1.2 $\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ \alpha_{\mathbf{L}\,c}$ *is a theorem.*
[Axiom of propositional variables.]

1.3 $\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ \mathbf{a}_{\mathbf{L}\,c}$ *is a theorem.*
[Axiom of semantical real variables.]

1.4 $\vDash [\mathbf{M\,N}]\,\supset\,=\ \alpha_{.0\,\mathbf{N}\,c}\,\beta_{.0\,\mathbf{N}\,c}\,[\mathbf{.0\,N}]\,\supset\,\alpha_{.0\,\mathbf{N}\,c}\,\beta_{.0\,\mathbf{N}\,c}$
[Logico-semantical axiom.]

1.5 $\vDash [\mathbf{M\,N}]\ \mathbf{Axsem}\ \mathbf{x}_{.0\,\mathbf{N}\,c}\,\mathbf{y}_{.0\,\mathbf{N}\,c}\,\mathbf{u}_{.0\,\mathbf{N}\,c}\,\mathbf{v}_{.0\,\mathbf{N}\,c}\,\mathbf{w}_{.0\,\mathbf{N}\,c}\,\mathbf{z}_{.0\,\mathbf{N}\,c}\,[\mathbf{.0\,N}]$
[Semantical axiom.]

Construction rules

2. Construction of expressions

2.1 If $\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ *\,*\ \mathbf{.I\,E\,K\,E\,I}$ and $\{\mathbf{M\,.0\,K}\}$ *are theorems,* then
$\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ *\,*\ \mathbf{.I\,.0\,K\,.0\,E\,I}$ *is a theorem.*
[Rule of propositional real variables.]

2.2 If $\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ *\,*\ \mathbf{.\Pi\,K\,E\,I}$ and $\{\mathbf{M\,.0\,K}\}$ *are theorems,* then
$\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ *\,*\ \mathbf{.\Pi\,.0\,K\,.0\,E\,I}$ *is a theorem.*
[Rule of semantical real variables.]

2.3 If $\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ *\,*\ \mathbf{J\,E\,I}$, *is a theorem,* then
$\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ *\,*\ \mathbf{J\,.0\,E\,I}$ *is a theorem.*
[The recursive rule of real variables.]

2.4 If $\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ \mathbf{E}$ and $\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ \mathbf{F}$ *are theorems,* then
$\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ *\ \mathbf{E\,F}$ *is a theorem.*
[Rule of the star-operator.]

3. Construction of propositions

3.1 If $\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ *\,*\ \mathbf{.I\,K\,E\,I}$ *is a theorem,* then
$\vDash [\mathbf{M\,N}]\ \mathbf{Prop}\ *\,*\ \mathbf{.I\,K\,E\,I}$ *is a theorem.*
[Rule of simple ambiguous propositions.]

3.2 If $\vDash [\mathbf{M\,N}]\ \mathbf{Expr}\,[\mathbf{L}]\ *\,*\,*\,*\ \mathbf{.0\,L\,E\,F\,G\,H}$ *is a theorem,* then
$\vDash [\mathbf{M\,N}]\ \mathbf{Prop}\,(\mathbf{E\,F\,G\,H})\,[\mathbf{L}]$ *is a theorem.*
[Rule of pattern of substitution.]

3.3 If **⊨ [M N] Prop F** and **gen [K N] (E J) F G** *are theorems,* then
⊨ [M N] Prop G *is a theorem.*
[Rule of general quantifiers.]

3.4 If **Expr N** and **⊨ [M .0 N] Prop E** *are theorems,* then
⊨ [M N] Prop E *is a theorem.*
[Rule of systems.]

3.5 If **⊨ [M N] Prop E** and **⊨ [M N] Prop F** *are theorems,* then
⊨ [M N] Prop / E F *is a theorem.*
[Rule of the stroke-operator.]

4. Rules of demonstration

4.1 If **⊨ [M N] Prop / / / E F G H** *is a theorem,* then
⊨ [M N] Ax log E F G H *is a theorem.*
[Nicod's syllogism.]

4.2 If **⊨ [M N] E** and **⊨ [M N] ⊃ E F** *are theorems,* then
⊨ [M N] F *is a theorem.*
[Modus ponens.]

4.3 If **∧ gen [K L] (E J) F G (F ** J E I X H)** and
⊨ [M N] ∧ Expr [K] X Prop / F G *are theorems,* then
(*a*) If **= J .I K** and **⊨ [M N] Prop X** *are theorems,* then
⊨ [M N] ⊃ G H *is a theorem.*
(*b*) If **= J .II K** *is a theorem,* **⊨ [M N] ⊃ G H** *is a theorem.*
[Deduction.]

4.4 If **∧ gen [K L] (E J) F G ~ {H ** J E I}**, **⊨ [M N] Prop G** and **⊨ [M N] ⊃ H F** *are theorems,* then
⊨ [M N] ⊃ H G *is a theorem.*
[Generalization.]

4.5 If **∧ (F**$\mathbf{x}_{\mathbf{K}\,0}$**.0KE) ∧ ((F**$\mathbf{x}_{\mathbf{K}\,0}\mathbf{y}_{\mathbf{K}\,0}$**G)) (F**$\mathbf{x}_{\mathbf{K}\,0}$*****$\mathbf{x}_{\mathbf{K}\,0}\mathbf{y}_{\mathbf{K}\,0}$**H)** and **⊨ [M N] ∧ E ⊃ ∧ F G H** *are theorems,* then
⊨ [M N] F *is a theorem.*
[Induction.]

It will be seen that where **M** is any fundamental integer greater than **N**, the proper system **[M N]** can be constructed.

The systems **[2 0]** and **[2 1]** will be discussed.

The application of axiom 1.1 to **[2 0]** yields the theorems :

(1) **⊨ [2 0] Expr [1] .0 1**
(2) **⊨ [2 0] Expr [2] .0 2**

These theorems state that **.0 1** is an expression of type **1** and **.0 2** is an expression of type **2**. These theorems can also be regarded as axioms of semantical identity.

If rule 2.4 is employed together with the theorems (1) and (2) as many expressions of types **1** and **2** as are desired may be obtained.

Axioms 1.2 and 1.3 involve real variables.

If **K** is any integer greater than **0**, $\alpha_{\mathbf{K}\,\mathbf{c}}$, $\beta_{\mathbf{K}\,\mathbf{c}}$, . . . $\mathbf{a}_{\mathbf{K}\,\mathbf{c}}$, $\mathbf{b}_{\mathbf{K}\,\mathbf{c}}$ . . . are **K**-type variables, the expressions $\alpha_{\mathbf{K}\,\mathbf{c}}$, $\beta_{\mathbf{K}\,\mathbf{c}}$ . . . are propositional real variables and the expressions $\mathbf{a}_{\mathbf{K}\,\mathbf{c}}$, $\mathbf{b}_{\mathbf{K}\,\mathbf{c}}$ . . . are semantical real variables. It is clear that these real variables are always original expressions of type **c**. The values of the semantical **K**-type variables are expressions of type **K**. The values of the propositional **K**-type variables are expressions of type **K** which are propositions.

From 1.2 and 1.3 it is possible to derive:

(3) $\models$ **[2 0] Expr [1]** $\alpha_{1\,\mathbf{c}}$
(4) $\models$ **[2 0] Expr [2]** $\alpha_{2\,\mathbf{c}}$
(5) $\models$ **[2 0] Expr [1]** $\mathbf{a}_{1\,\mathbf{c}}$
(6) $\models$ **[2 0] Expr [2]** $\mathbf{a}_{2\,\mathbf{c}}$

These theorems state that any values of $\alpha_{1\,\mathbf{c}}$ and $\mathbf{a}_{1\,\mathbf{c}}$ are expressions of type **1**, and any values of $\alpha_{2\,\mathbf{c}}$ and $\mathbf{a}_{2\,\mathbf{c}}$ are expressions of type **2**.

The difference in the interpretation of these theorems and theorems (1) and (2) is obvious since here original expressions of type **c** which cannot be discussed by means of the patterns **(E F G H) [1]** and **(E F G H) [2]** are involved.

These theorems and the rules 2.1, 2.2, and 2.3 can be used to prove the theorems:

(7) $\models$ **[2 0] Expr [1]** $\alpha_{2\,\mathbf{c}}$
(8) $\models$ **[2 0] Expr [1]** $\mathbf{a}_{2\,\mathbf{c}}$
(9) $\models$ **[2 0] Expr [1]** $\beta_{1\,\mathbf{c}}$ and so on.

These theorems together with 2.4 will yield as many variable expressions of types **1** and **2** as are desired. For example, the following theorems can be derived:

(10) $\models$ **[2 0] Expr [1]** $* \, \mathbf{x}_{1\,\mathbf{c}} \mathbf{y}_{1\,\mathbf{c}}$

(11) $\models$ **[2 0] Expr [1]** $* * * * \, \mathbf{.01} * \mathbf{x}_{1\,\mathbf{c}} \mathbf{y}_{1\,\mathbf{c}} \mathbf{u}_{1\,\mathbf{c}} \mathbf{v}_{1\,\mathbf{c}} * \mathbf{w}_{1\,\mathbf{c}} \mathbf{z}_{1\,\mathbf{c}}$

Theorem (10) states that for any values of $\mathbf{x}_{1\,\mathbf{c}}$, $\mathbf{y}_{1\,\mathbf{c}}$, the expression $* \, \mathbf{x}_{1\,\mathbf{c}} \mathbf{y}_{1\,\mathbf{c}}$ is an expression of type **1**.

Theorem (11) states that for any values of $\mathbf{x}_{1\,\mathbf{c}}$, $\mathbf{y}_{1\,\mathbf{c}}$, $\mathbf{u}_{1\,\mathbf{c}}$, $\mathbf{v}_{1\,\mathbf{c}}$, $\mathbf{w}_{1\,\mathbf{c}}$, and $\mathbf{z}_{1\,\mathbf{c}}$ the expression

$$* * * * .\mathbf{0}\,\mathbf{1} * \mathbf{x}_{1\,\mathbf{c}}\mathbf{y}_{1\,\mathbf{c}}\mathbf{u}_{1\,\mathbf{c}}\mathbf{v}_{1\,\mathbf{c}} * \mathbf{w}_{1\,\mathbf{c}}\mathbf{z}_{1\,\mathbf{c}}$$

is an expression of type **1**.

It should be noted that the construction of variables which has been given enables us to see at once, whether a given expression is a constant or a variable expression of type **K**. In the first case it does not contain **I** or **II**. In the second it must contain either **I** or **II**.

It should also be noted that no individual property of a real variable can be proved in a proper system, because its components **I** or **II** are original expressions of type **c** which cannot be expressions of a proper system.

The rules of propositions permit the derivation of the following theorems :

$$\models [\,2\,0\,]\ \mathbf{Prop}\,\alpha_{1\,\mathbf{c}}$$
$$\models [\,2\,0\,]\ \mathbf{Prop}\,\beta_{1\,\mathbf{c}}$$
$$\cdots\cdots\cdots$$
$$\models [\,2\,0\,]\ \mathbf{Prop}\,\alpha_{2\,\mathbf{c}}$$
$$\cdots\cdots\cdots$$

These theorems state that any value of a propositional real variable is a proposition.

The application of rule 3.2 yields the theorems :

$$\models [\,2\,0\,]\ \mathbf{Prop} = \alpha_{1\,\mathbf{c}}\,\beta_{1\,\mathbf{c}}\,[\,1\,]$$
$$\models [\,2\,0\,]\ \mathbf{Prop}\,(\mathbf{x}_{1\,\mathbf{c}}\mathbf{y}_{1\,\mathbf{c}}\mathbf{u}_{1\,\mathbf{c}}\mathbf{v}_{1\,\mathbf{c}})\,[\,1\,]$$
$$\models [\,2\,0\,]\ \mathbf{Prop} = \mathbf{x}_{1\,\mathbf{c}}\ .\mathbf{0}\,\mathbf{1}\,[\,1\,]$$
$$\models [\,2\,0\,]\ \mathbf{Prop} = \mathbf{v}_{1\,\mathbf{c}}.\mathbf{0}\,\mathbf{1}\,[\,1\,]$$
$$\models [\,2\,0\,]\ \mathbf{Prop} = .\mathbf{0}\,\mathbf{1} * \mathbf{x}_{1\,\mathbf{c}}\mathbf{y}_{1\,\mathbf{c}}\,[\,1\,]$$
$$\models [\,2\,0\,]\ \mathbf{Prop}\,(\mathbf{x}_{1\,\mathbf{c}}\mathbf{y}_{1\,\mathbf{c}}\mathbf{u}_{1\,\mathbf{c}}\mathbf{v}_{1\,\mathbf{c}}\mathbf{w}_{1\,\mathbf{c}}\mathbf{z}_{1\,\mathbf{c}})\,[\,1\,]$$
$$\cdots\cdots\cdots\cdots\cdots$$

Now with the help of 3.5

$$\models [\,2\,0\,]\ \mathbf{Prop}\,\supset\, = \alpha_{1\,\mathbf{c}}\,\beta_{1\,\mathbf{c}}\,[\,1\,]\,\supset \alpha_{1\,\mathbf{c}}\,\beta_{1\,\mathbf{c}}$$
$$\models [\,2\,0\,]\ \mathbf{Prop\ Ax\ sem}\,\mathbf{x}_{1\,\mathbf{c}}\mathbf{y}_{1\,\mathbf{c}}\mathbf{u}_{1\,\mathbf{c}}\mathbf{v}_{1\,\mathbf{c}}\mathbf{w}_{1\,\mathbf{c}}\mathbf{z}_{1\,\mathbf{c}}\,[\,1\,]$$

are proved.

To obtain general quantifiers 3.4 will be employed.

We have the theorem

$$\mathbf{gen}\,[\,2\,1\,]\ (\mathbf{1}_2\,.\mathbf{I}\,\mathbf{2}) \sim \alpha_{2\,\mathbf{c}}\,\mathbf{II}\,[\,2\,1\,]\ \alpha_{2\,1} \sim \alpha_{2\,1}$$

Then by the application of rule 3.3, it is possible to obtain the theorem :

$$\models [\,2\,1\,]\ \mathbf{Prop}\,\mathbf{II}\,[\,2\,1\,]\ \alpha_{2\,1} \sim \alpha_{2\,1}$$

From this theorem, the application of rule 3.4 yields the theorem :

$$\models \mathbf{[20]\ Prop\ \Pi[21]}\ \alpha_{2\,1} \sim \alpha_{2\,1}$$

Then by the application of 3.5 the theorem :

$$\models \mathbf{[20]\ Prop\ \exists[21]}\ \alpha_{2\,1}\,\alpha_{2\,1}$$

will be obtained.

It can be seen at once that

$$\mathbf{gen[20]\,(1_2\,.\Pi 2) = a_{2\,0}a_{2\,0}[1]\,\Pi\,[20]\,a_{2\,0} = a_{2\,0}\,a_{2\,0}[1]}$$

is a theorem. Thus by 3.3

$\models \mathbf{[20]\ Prop\ \Pi[20]\ a_{2\,0} = a_{2\,0}\ a_{2\,0}[1]}$ *is a theorem.*

With the quantifiers the list of the primitive propositions which occur in the proper systems is completed. In proper systems there are

1. Patterns of substitution.
2. Sheffer's strokes and quantifiers.

If a pattern of substitution or a Sheffer's stroke is a proposition of a proper system, it may be interpreted in the same way as were the corresponding propositions of the auxiliary system.

If a pattern $\mathbf{\Pi[KN]XF}$ is a proposition of a proper system, then the proposition : **All values of $\mathbf{\Pi[KN]XF}$ are true symbolic propositions** is the interpretation of **$\mathbf{\Pi[KN]XF}$ is a true symbolic proposition.**

It is clear that a value of $\mathbf{\Pi[21]}\,\alpha_{2\,1} \sim \alpha_{2\,1}$ is any expression of type **2** which is a proposition, and a value of $\mathbf{\Pi[20]\,a_{2\,0} = a_{2\,0}\,a_{2\,0}[1]}$ is any proposition $\mathbf{= E\,E[1]}$ where **E** is any expression of type **2**.

It should be noted that the propositions $\mathbf{\Pi[MN]XF}$ are never expressions of $\mathbf{[MN]}$. For example $\mathbf{\Pi\,[21]}\alpha_{2\,1} \sim \alpha_{2\,1}$ is not an expression of $\mathbf{[21]}$ but is an expression of $\mathbf{[20]}$. Consequently it cannot be discussed in $\mathbf{[21]}$ but it can be the object of metamathematical research in $\mathbf{[20]}$. This fundamental property of the quantifiers prevents the occurrence of vicious circle fallacies in our proper systems.

The logico-semantical axiom is required to construct the theory of classes and relations. It should be noted that although it can be proved for any particular case, its contradictory is consistent with the other axioms and rules of the proper systems. If its contradictory were assumed an idealistic system would be obtained which would be of the same type as Plato's mythology and which as a matter of fact is consistent.

The logico-semantical axiom permits the fundamental laws of identity to be proved in simple fashion, but Hetper obtained the same result by employing the rule of induction. He proved that there are simple systems **[.0NN]** $_H$ which contain all the axioms and rules of **[.0NN]** with the exception of those which contain propositional variables. These systems can be employed for the construction of rational arithmetic and also for the fundamental meta-mathematical researches. This remark will be amplified below.[1]

With the exception of the rule of induction the rules (*R P*) of demonstration conform to those of the ordinary calculus of quantifiers. Since the rule of induction is a simple description of semantical facts, here it is as general as the other fundamental rules. It is therefore not a synthetic apriori judgment.

5. If **Λ IntegL Λ (N0.0LM)(E0.0LF)** and $\models$ **[N0]E**, *are theorems*, then **F** *is a theorem* of the *metasystem* **(M.1L)**. The corresponding symbolic proposition will be constructed in the system **[.0LL]**. It will be abbreviated by the symbol:

$$\vdash_L \mathbf{(M.1L)F}$$

It should be noted that the same symbol will be employed if a construction is made in the meta-system **(.0LL)**. If our proposition is true, it states that **F** is an element **Z** of a recursive finite class, whose initial member

$$\mathbf{AxEF(M.1LL)Z}$$

determines the axiom, and whose recursive member

$$\mathbf{DEFGHJK(M.1LL)XYZ}$$

determines the theorems of **(M.1L)**.

This construction is based upon Hetper's concept of intervals[2] and a remark of Pepis. It may be found in *N. F. F. M.*[3]

The theory of metasystems and Hetper's theory of simple metasystems permits a formal discussion of the decision

[1] Cf. W. Hetper: "Simple Systems, Meta-systems, and Generalized Meta-systems" (unpublished).

[2] Cf. W. Hetper: "Relacje ancestralne w systemie semantyki" ("Ancestral Relations in the System of Semantics", *Archiwum Towarzystwa naukowego we Lwowie*, Dział III, Tom IX, 1938, pp. 265–280.

[3] Cf. *N. F. F. M.*, p. 35.

problem. It is known that in the elementary system of the logical calculus it can be decided whether or not a given propositional function is a theorem of this system. Hetper proved that a similar problem can be handled in the case of an elementary system of semantics, on the basis of the relation $\{\mathbf{E}\,\mathfrak{F}\}$, where the system contains no apparent variables.[1] The question has also been raised whether it is possible to construct a system of mathematics in which every question that can be formulated would be decidable. When I was working in the domain of pure theory of types, a sphere where, of necessity, many questions are undecidable, I always regarded with great scepticism the efforts made to show that the propositions of mathematics are decidable. My position was completely confirmed by the work of Gödel who proved that in any system which contains the system of natural numbers, it is not possible to speak about the decidability of all problems.[2] Gödel had to expend great effort to obtain his result, and had to create *ad hoc* intuitive semantics. It is obvious that matters of this kind can be investigated very easily with the help of the theory of metasystems.

If the simple meta-system $(3\,2)$ is considered, it can be supposed to be a simple metasystem of Hetper and therefore cannot contain classes. The following theorems can be derived.[3]

$$\models [1\,0] \supset\, \vdash_0 (3\,2)\, t_{1\,0}\, \exists [1\,0]\, s_{1\,0} \wedge (t_{1\,0}\, 0_1\, 2_1\, s_{1\,0})\, [1] \\ \vdash_0 (3\,2) \vdash_2 (5\,4)\, s_{1\,0}$$

$$\models [1\,0] \vdash_0 (3\,2) \supset\, \vdash_2 (5\,4)\, t_{1\,3}\, \exists [3\,2]\, s_{3\,2} \wedge (t_{3\,1}\, 0_3\, 2_3\, s_{3\,2})\, [3] \\ \vdash_2 (5\,4) \vdash_4 (7\,6)\, s_{3\,2}$$

Now we assume the following construction :

$G_L(E)$	is an abbreviation of	$\exists\, [\,.0\, L\, L\,]\, \bar{x}_L\, \bar{y}_L\, \bar{z}_L\, \bar{u}_L \wedge = E\, \Pi\, [\,.2\, L\, .1\, L\,]\, \bar{y}_L\, \bar{z}_L\, [\,.0\, L\,] \wedge (E\, .0\, 0_L\, .0\, 2_L\, \bar{x}_L)\, [\,.0\, L\,] \wedge (\bar{z}_L\, \bar{\bar{y}}_L\, \bar{x}_L\, \bar{u}_L)\, [\,.0\, L\,] \sim\, \vdash_L (\,.2\, L\, .1_L\,)\, \bar{\bar{u}}_L$
$G_{(L)}$		$G_L(\Pi\, [\,.2\, L\, .1\, L\,]\, \bar{a}_{.1\,L}\, G_{.1\,L}(\bar{a}_{.1\,L}))$

[1] W. Hetper : " Zagadnienie zupełności systemu elementarnej semantyki " (" The problem of the Completeness of the System of Elementary Semantics "), *l.c.*

[2] Kurt Gödel : " Über formal unentscheidbare Sätze der Principia Mathematica, etc.," *Monatschefte für Mathematik und Physik*, Bd. 38, 1931, pp. 173–198.

[3] Cf. *F. P. G. T.*

The following theorems can be derived :

$$\models [10] \supset \vee \vdash_0 (32) G_{(2)} \vdash_0 (32) \sim G_{(2)} \vdash_0 (32) \vdash_2 (54] = 0_5 1_5 [5]$$

$$\models [10] \supset \vdash_0 (32) \sim \vdash_2 (54) = 0_5 1_5 [5] \vdash_0 (32) = 0_3 1_3 [3]$$

These are Gödel's theorems. It is obvious that the proposition $G_{(2)}$ is not decidable in the metasystem (32). If it were it could be proved that the metasystem (54) contains a contradiction. It can also be seen that it is not possible to prove the consistency of (54) in the metasystem (32). If this were possible this metasystem would contain a contradiction.

If it is now supposed that the theorem :

$$\models [10] \sim \vdash_0 (32) = 0_3 1_3 [3]$$

has been proved, step by step using the same method, it would be possible to prove the theorem :

$$\models [10] \vdash_0 (32) \sim \vdash_2 (54) = 0_5 1_5 [5]$$

From this theorem and from Gödel's theorem, the theorem

$$\models [10] \vdash_0 (32) = 0_3 1_3 [3]$$

can be derived. Then we have the theorem :

$$\models [10] \wedge \sim \vdash_0 (32) = 0_3 1_3 [3] \vdash_0 (32) = 0_3 1_3 [3]$$

It is obvious that our hypothesis implies a contradiction in the simple system $[10]$.

6. The following abbreviations will be employed in developing the elementary theory of classes and relations :

Subst $[ML]EFGXH$	is an abbreviation of	$\wedge = F \,\Pi [M.0L] E G [.0L]$ $\wedge \{GE\} [.0L] (GEXH) [.0L]$
Class $[ML]F$		$\Pi [ML] x_{ML} \exists [.1LL] \bar{\bar{a}}_L \exists [.0LL] \bar{g}_L$ Subst $[ML] a_{M.0L} F \bar{g}_L x_{ML} \bar{\bar{a}}_L$
$\varepsilon [ML] XF$		$\exists [.1LL] \bar{\bar{o}}_L \wedge \bar{\bar{a}}_L \exists [.0LL] \bar{g}_L$ Subst $[ML] a_{M.0L} F \bar{g}_L X \bar{\bar{a}}_L$
Relat $[NML]E$		$\Pi [NL] y_{NL} \exists [.0LL] \bar{h}_L$ $\wedge$ Class $[ML] \bar{h}_L \exists [.0LL] \bar{f}_L$ Subst $[ML] b_{N.0L} E \bar{f}_L y_{NL} \bar{h}_L$
rel $[NML]E(XY)$		$\exists [.0LL] \bar{h}_L \wedge \varepsilon [ML] Y \bar{h}_L \exists [.0L] \bar{f}_L$ Subst $[NL] b_{N.0L} E \bar{f}_L X \bar{h}_L$

These symbols are identical with those which were introduced in Chapter VI, but in addition the types have been indicated.

If any proposition $\mathbf{\Pi[M.0L]\,a_{M.0L}\,F}$ is given, it can be shown that it is a class. If any proposition $\mathbf{\Pi[N.0L]\,b_{N.0L}\,\Pi[M.0L]\,a_{M.0L}\,F}$ is given, it can be shown that it is a relation between expressions of type **N** and expressions of type **M**. **.0 L** is the type of our classes and our relations.

It is now possible to prove the following theorems :

$$\models [20] \equiv \mathrm{Expr}[2]\,x_{2\,c}\,\varepsilon[20]\,x_{2\,c}\,\Pi[21]\,a_{2\,1}\,\mathrm{Expr}[2]\,a_{2\,1}$$

$$\models [20] \equiv \mathrm{Integ}[2]\,x_{2\,c}\,\varepsilon[20]\,x_{2\,c}\,\Pi[21]\,a_{2\,1}\,\mathrm{Integ}[2]\,a_{2\,1}$$

$$\models [30] \equiv \varepsilon[42]\,x_{4\,c}\,f_{3\,c}\,\mathrm{rel}[430]\,\Pi[31]\,b_{3\,1}\,\Pi[41]\,a_{4\,1}\\ \varepsilon[42]\,a_{4\,1}\,b_{3\,1}\,(f_{3\,c}\,x_{4\,c})$$

It is clear that there is a complete analogy between our constructions and the primitive idea of a class.

It is now possible to build up sums, products, and the complementary classes of given classes by employing classical constructions.

$\mathbf{-[ML]E}$	is an abbreviation of	$\mathbf{\Pi[ML]\,a_{ML}\sim\varepsilon[M.0L]\,a_{KL}\,E}$
$\mathbf{\frown[ML]EF}$		$\mathbf{\Pi[ML]\,a_{ML}\wedge\varepsilon[M.0L]\,a_{ML}\,E\,\varepsilon[M.0L]\,a_{\ L}\,F}$
$\mathbf{\smile[ML]EF}$		$\mathbf{\Pi[ML]\,a_{ML}\vee\varepsilon[M.0L]\,a_{ML}\,E\,\varepsilon[M.0L]\,a_{ML}\,F}$

These are the patterns of the complementary class of **E** and the product of the classes **E** and **F** and the sum of the classes **E** and **F**. It can be proved that there exist classes of type **.1 L**, i.e. of the same type as the classes **E** and **F** which are equal to their complementary classes and to their products and sums. The demonstration of these theorems is based upon the logico-semantical axiom. The postulates of Huntington can now be proved in a very simple manner. Analogous results may be obtained for relations.[1]

It is evident that the same method can be applied in the

[1] Cf. *N. F. F. M.*, VI, 2.

case of relations having any number of members. Consequently it is possible to construct matrices, tensors, etc., in a very simple way.

It should be noted that our classes and relations are general propositions, but their truth-values are never considered because it is desired to avoid new fundamental patterns. If their truth-values were considered a simplification in the use of types would result.[1]

Our theory of classes permits the introduction of real numbers. Given any two real numbers of the same type it can be proved that their sum and product is a number of the same type. Nevertheless we cannot obtain a general theorem on this subject without employing Hetper's generalized system, which will be introduced later.

With this theory of real numbers it is possible to build up an elementary theory of functions of real variables in the meta-systems. An incomplete continuum more or less like that constructed by H. Weyl[2] is involved here.

To see how Richard's paradox is avoided, it should be observed that here as well as in the pure theory of types there are orders of classes.

The difference between the type of a class and the type of its elements will be called the *order* of this class.

The following construction will be assumed.

$\mathbf{C}_{(\mathbf{L})}$	is an abbreviation of	$\Pi\,[.3\,L\,L]\,a_{.3\,L\,L}\,\exists\,[.1\,L\,.0\,L]\,\bar{\bar{x}}_{.0\,L} \wedge \mathrm{Class}\,[.3\,L.0\,L]\,\bar{\bar{x}}_{.0\,L} \wedge (\bar{\bar{x}}_{.0\,L}\,.0\,1_L\,.0\,3_L\,a_{.3\,L\,L})\,[.1\,L] \sim \varepsilon\,[.3\,L\,.0\,L]\,a_{.3\,L\,L}\,\bar{\bar{x}}_{.0\,L}$

It is clear that $\mathbf{C}_{(3)}$ is the class of classes of type **5** whose elements are expressions of type **7** intertypically different from them. $\mathbf{C}_{(3)}$ is a class of order **− 4**. It can be shown that it is different from all classes of order **− 2**. As a matter of fact, given any class **E** of order **− 2**, where **E′** is the expression of type **7** which is intertypically equal to **E**, two cases are possible. If **E′** is an element of **E**, it is not an element of $\mathbf{C}_{(3)}$; if it is not an element of **E**, it is an element of $\mathbf{C}_{(3)}$. Then $\mathbf{C}_{(3)}$ cannot have the same elements as **E**. It is clear that $\mathbf{C}_{(3)}$ is strictly analogous to the number on

[1] Cf. J. Herzberg: "Sur la notion de collectif," *Annales de la Société Polonaise de Mathématique*, tom xvii, 1938, pp. 232 f.

[2] Cf. H. Weyl: *Das Kontinuum*, Leipzig, 1918

the diagonal used by Cantor to prove that the continuum is a non-denumerable class. The class $\mathbf{C}_{(3)}$ is of fundamental importance for our researches on the reconstruction of Cantor's theory. A brief account of these investigations will be given below.

It should be noted that a discussion of the antinomies based upon the idea of a class or relation is as a matter of fact superfluous because if there were a contradiction in our calculus of classes or relations it would be a contradiction of the simple semantical calculus. It should be observed that Russell's paradox about classes which are not elements of themselves can be introduced. The proposition

$$\varepsilon[\mathbf{1\,0}]\,\Pi[\mathbf{2\,1}]\,\mathbf{a}_{2\,1}\,\mathbf{f}_{2\,c}\,\Pi[\mathbf{2\,1}]\,\mathbf{a}_{2\,1}\,\mathbf{f}_{2\,c}$$

and the proposition

$$\varepsilon[\mathbf{1\,0}]\,\Pi[\mathbf{2\,1}]\,\mathbf{a}_{2\,1}\,\mathbf{f}_{2\,c}\,\Pi[\mathbf{2\,1}]\,\mathbf{a}_{2\,1}\sim\mathbf{f}_{2\,c}$$

can be shown to be always false.

7. We begin with *the system* $(\mathbf{L})_\infty$ *of rational metamathematics.* For this purpose we will employ the pattern

$$\exists[\mathbf{.0\,L\,L}]\,\bar{\mathbf{m}}_{\mathbf{L}}\vdash_{\mathbf{L}}(\bar{\mathbf{m}}_{\mathbf{L}}\mathbf{.1\,L})\,\mathbf{E}$$

which will be abbreviated by $\vdash_{\mathbf{L}}\mathbf{E}$.

If $\vdash_{\mathbf{L}}\mathbf{E}$ is a true proposition of $[\mathbf{1\,0}]$, then $\mathbf{E}$ must be a true proposition of a meta-system $(\mathbf{M\,.1\,L})$. It is obvious that this construction permits the discussion of all types of expressions which contain $\mathbf{.0\,.1\,L}$. Thus in the system $(\mathbf{0})_\infty$ we have all types of expressions which contain $\mathbf{3}$, i.e. the numbers

$$\begin{array}{l} \mathbf{0}_3,\ \mathbf{1}_3,\ \mathbf{2}_3,\ \mathbf{3}_3,\ \ldots \\ \quad\mathbf{0}_4,\ \mathbf{1}_4,\ \mathbf{2}_4,\ \ldots \\ \qquad\mathbf{0}_5,\ \mathbf{1}_5,\ \ldots \\ \qquad\ldots\ldots \end{array}$$

It is clear that the number $\mathbf{n}$ of type $\mathbf{m}$ equals the number $\mathbf{n}-\mathbf{1}$ of type $\mathbf{m}+\mathbf{1}$. There exists a very simple relation between equal numbers of different types which can be called *intertypical equality.* To construct this relation the symbol $=(\mathbf{G\,H})\,\mathbf{E\,F}[\mathbf{K}]$ will be introduced. This expression is an abbreviation of

$$\wedge\,(\mathbf{E\,.0\,G\,.0\,H\,F})[\mathbf{K}]\,\wedge=\mathbf{E\,E}[\mathbf{G}]=\mathbf{F\,F}[\mathbf{H}].$$

If $=(\mathbf{G\,H})\,\mathbf{E\,F}[\mathbf{K}]$ is a true proposition of $(\mathbf{0})_\infty$, $\mathbf{E}$ and $\mathbf{F}$ will be said to be intertypically equal with regard to the types $\mathbf{G}$ and $\mathbf{H}$.

Now :

$$\models [10] \vdash_0 = (34)45[3]$$
$$\models [10] \vdash_0 = (35)46[3]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\models [10] \vdash_0 = (34)56[3]$$
$$\models [10] \vdash_0 = (35)57[3]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

It might be thought that two intertypically equal classes have intertypically equal elements. It will be seen that this is not the case. Let the following construction be considered :

$\mathbf{R}_{(L)}$	is an abbreviation of	$\Pi[.1LL]\,\bar{\bar{a}}_L \wedge \mathrm{Class}[.3L.0L]\,\bar{\bar{a}}_L \wedge \exists\,[.3L.0L]\,x_{.3L.0L} \wedge (\bar{\bar{a}}_L\,.0\,1_L\,.0\,3_L\,x_{.3L.0L})\,[.1L] \sim \varepsilon[.3L.0L]\,x_{.3L.0L}\,\bar{\bar{\bar{a}}}_L$

By an elementary calculation the following theorem can be proved :

$$\models [10] \vdash_0 \equiv \varepsilon[52]\,\mathbf{R}_{(5)}\,\mathbf{R}_{(3)} \sim \varepsilon[74]\,\mathbf{R}_{(7)}\,\mathbf{R}_{(5)}$$

It is clear that the classes $\mathbf{R}_{(5)}$ and $\mathbf{R}_{(3)}$ and the classes $\mathbf{R}_{(7)}$ and $\mathbf{R}_{(5)}$ are intertypically equal. Yet if $\mathbf{R}_{(5)}$ is an element of $\mathbf{R}_{(3)}$, $\mathbf{R}_{(7)}$ cannot be an element of $\mathbf{R}_{(5)}$ and vice versa.

This construction is very instructive. It can be seen that just as $\mathbf{G}_{(2)}$ was undecidable, so is the proposition $\varepsilon[52]\,\mathbf{R}_{(5)}\,\mathbf{R}_{(3)}$. But there is an essential difference between them. If $\mathbf{G}_{(2)}$ were a false proposition, presumably it would be a decidable proposition. Now $\mathbf{G}_{(2)}$ cannot be decidable unless (32) contains a contradiction. As it is not assumed that (32) is a contradictory system, it must be assumed that $\mathbf{G}_{(2)}$ is a true proposition of (32). We will say that while $\mathbf{G}_{(2)}$ is not a decidable proposition of (32), it is a *discutable* one. It will be seen that $\varepsilon[52]\,\mathbf{R}_{(5)}\,\mathbf{R}_{(3)}$ is not a discutable proposition. Any discussion of this proposition involves an infinite regress. Since the propositions $\varepsilon[52]\,\mathbf{R}_{(5)}\,\mathbf{R}_{(3)}$ and $\varepsilon[74]\,\mathbf{R}_{(7)}\,\mathbf{R}_{(5)}$ are entirely analogous, they are equiponderant. However, they are contradictory. Then there is a complete symmetry between the truth and falsehood of $\varepsilon[52]\,\mathbf{R}_{(5)}\,\mathbf{R}_{(3)}$.

The following construction rules can be assumed for equal classes:

I. *Two classes of expressions are equal if their elements are intertypically equal.*

II. *Two classes of classes are equal if their elements are equal classes.*

In the metasystem $\mathbf{(M\,N)}$ the quantifier $\mathbf{\Pi[M\,K\,J]\,X\,F}$ could be employed instead of $\mathbf{\Pi[K\,J]\,X\,F}$ to eliminate propositions which are not discutable.[1] In this way any metasystem $\mathbf{(M\,0)}$ would have separate general propositions and $\mathbf{R}_{(3)}$ and $\mathbf{R}_{(5)}$ would not be strictly analogous. If this theory were employed, it would be possible to avoid the residues of idealism which are implied by our method. Nevertheless it is of interest not to limit our domain of research in this way and so arrive at a reconstruction of the fundamental concepts of the general theory of classes.

The pattern :

$$\mathbf{expr_L\,[M]\,X}$$

will be employed. It is a Hetper interval which defines constant expressions of type M. I omit its construction. The following abbreviations will be posited :

$\mathbf{Expr\,(L)}$	is an abbreviation of	$\mathbf{\Pi\,[.1\,L\,L]\,\bar{\bar{\bar{b}}}_L\,\bar{\bar{a}}_L\,expr_{.0\,L}\,[\bar{\bar{\bar{b}}}_L]\,\bar{\bar{a}}_L}$
$\mathbf{typ_L\,(M\,N\,X)}$		$\mathbf{\wedge\,Integ\,[.0\,L]\,M\,\wedge\,\{M\,.1\,N\}\,[.0\,L]\,expr_L\,[.0\,N]\,X}$
$\mathbf{Inv_L\,U}$		$\mathbf{\Pi\,[.1\,L\,L]\,\bar{\bar{c}}_L\,\bar{\bar{\bar{b}}}_L\,\bar{\bar{a}}_L\,\wedge\,typ_{.0\,L}\,(\bar{\bar{c}}_L\,\bar{\bar{\bar{b}}}_L\,\bar{\bar{a}}_L)}$ $\mathbf{\exists\,[.1\,L\,.0\,L]\,\bar{a}_{.0\,L}\,\wedge\,\bar{a}_{.0\,L} = \bar{a}_{.0\,L}\,U\,[.1\,L]}$
$\mathbf{Cl\,(L)}$		$\mathbf{Inv_L\,Class\,[\bar{\bar{c}}_L\,\bar{\bar{b}}_L]\,\bar{\bar{a}}_L}$
$\mathbf{un\,[M\,L]\,E}$		$\mathbf{Class\,[M\,L]\,E\,\exists\,[M\,L]\,u_{M\,L}\,\Pi\,[M\,L]\,v_{ML}}$ $\mathbf{\equiv\,\varepsilon\,[M\,L]\,v_{M\,L}\,E = u_{M\,L}\,v_{M\,L}\,[M]}$
$\mathbf{Un\,(L)}$		$\mathbf{Inv_L\,un\,[\bar{\bar{c}}_L\,\bar{\bar{b}}_L]\,\bar{\bar{a}}_L}$

$\mathbf{Expr\,(9)}$ is the relation between types and the constant expressions belonging to their domain.

It is the *fundamental invariant of expressions.*

It corresponds to the primitive idea of the class of expressions.

$\mathbf{Cl\,(7)}$ is the relation between types of expressions, types of descriptions of classes, and the corresponding classes.

[1] *K. P. Z.*, p. 330.

It is the *fundamental invariant of simple classes.*

It corresponds to the primitive idea of the class of classes.

$\mathbf{Cl}_{\mathrm{II}}$ (L), i.e. the fundamental relation of classes of classes, **$\mathbf{Cl}_{\mathrm{III}}$ (L)** the fundamental relation of classes of classes of classes, etc., can be constructed analogously.

It may also be remarked that **Un (7)** is the relation between types of expressions, types of descriptions of classes, and the corresponding unit classes.

It is an invariant of the same degree as **Cl (7)**.

It should be noted that the use of $\bar{\bar{b}}_9$, $\bar{\bar{c}}_7$, and $\bar{\bar{b}}_7$ is quite different from the use of $\bar{\bar{a}}_9$ and $\bar{\bar{a}}_7$. In the first case we have *type-parameters*, in the second *class-parameters*.

The value of **Expr (9)** which corresponds to any value of the type-parameter $\bar{\bar{b}}_9$ is either a class of expressions of the given type or a null class. In the first case it is the class-value of **Expr (9)**. The corresponding value of $\bar{\bar{b}}_9$ is its *proper value.*

The same concepts can be applied to any invariant. For example, the class-value of **Cl (7)** for the values $\overline{12}$ and $\overline{10}$ of its type-parameters is the class ;

$$\Pi[97]\bar{\bar{a}}_7 \wedge \mathrm{typ}_8(\overline{12}\,\overline{10}\,\bar{\bar{a}}_7)\exists[98]\alpha_8 \wedge \alpha_8 = \alpha_8\,\mathrm{Class}[\overline{12}\,\overline{10}]\bar{\bar{a}}_7[9]$$

This class is equal to the class

$$\Pi[\overline{11}\,\overline{10}]\,\bar{a}_{10}\ \mathrm{Class}[\overline{12}\,\overline{10}]\,\bar{a}_{10}$$

It can be seen that it is the class of classes of type $\overline{11}$ of expressions of the type $\overline{12}$.

If the same proper values of $\bar{\bar{c}}_7$ and $\bar{\bar{b}}_7$ are taken in **Un (7)** and **Cl (7)**, two class-values of our invariants are obtained. The first is a sub-class of the other. **Un (7)** is said to be a sub-invariant of **Cl (7)**. It can be seen that any invariant is a sub-invariant of the fundamental invariant of the same degree. A one to one relation between two invariants will now be defined.

If **Un (7)** and **Expr (9)** are considered, to any element **E** of a given class-value of **Un (7)** belongs as relatum in any class-value of **Expr (9)** the expression **X** which is intertypically equal to the element of **E**.

Likewise to any element **X** of a given class-value of **Expr (9)** belongs as referent in any class value of **Un (7)** any unit class whose element is intertypically equal to **X**.

We may therefore say that there is a one to one relation between **Un (7)** and **Expr (9)** and these invariants can be called *similar.*

It has been seen that there is a sub-invariant of **Cl (7)** which is similar to **Expr (9)**. It will now be proved that there is no sub-invariant of **Expr (9)** which is similar to **Cl (7)**.

If it is supposed that **J** is such an invariant, then an expression **X** of type $\overline{\mathbf{13}}$ which is an element of the class-value of **J** corresponding to the value $\overline{\mathbf{13}}$ of $\bar{\bar{\mathfrak{b}}}_9$, should belong as relatum to Cantor's class $\mathbf{C}_{(\,9\,)}$.

The same expression should be the relatum of a class **E** of type $\overline{\mathbf{11}}$, which is an element of the class-value of **Cl (7)** corresponding to the values $\overline{\mathbf{13}}$ and **9** of its type-parameters $\bar{\bar{\mathfrak{c}}}_7$, and $\bar{\bar{\mathfrak{b}}}_7$. Now since $\mathbf{C}_{(\,9\,)}$ is different from all classes of type $\overline{\mathbf{11}}$, **X** should have two different classes as referents in two different class-values of **Cl (7)**. Consequently a contradiction is obtained and it is said that no sub-invariant of **Expr (9)** can be similar to **Cl (7)**.

Let it now be supposed that the invariant **E** which is a sub-invariant of **F** is similar to **G**. *The cardinal number of* **E** is said to *be greater than or equal to the cardinal number of* **G**.

If this relation holds between **E** and **F** as well as between **F** and **E**, **E** and **F** are said to *have equal cardinal numbers.*

If this relation holds between **E** and **F** but not between **F** and **E** it is said that *the cardinal number of* **E** *is greater than the cardinal number of* **F**.

It can be seen that the cardinal number of **Cl (7)** is greater than the cardinal number of **Expr (9)**, and that this result does not depend upon the types **7** and **9** which have been used.

Our theory of cardinal numbers, therefore, seems to conform to Cantor's fundamental ideas, but is more complicated. To construct it it is necessary to begin with two infinite systems $(\mathbf{0})_\infty$ and $(\mathbf{2})_\infty$. We have, for example, the following theorems:

(1) $\models [\,1\,0\,] \vdash_0 \supset \mathbf{typ}_8\,(\,\mathbf{m}_{9\,1}\,\mathbf{n}_{9\,1}\,\mathbf{x}_{9\,1}\,)$
$\vdash_2 \equiv \mathbf{rel}[9996]\,\mathbf{Cl}(7)\,(\mathbf{m}_{9\,1}\mathbf{n}_{9\,1}\mathbf{x}_{9\,1})\,\mathbf{Class}[\mathbf{m}_{9\,1}\mathbf{n}_{9\,1}]\,\mathbf{x}_{9\,1}$

(2) $\models [\,1\,0\,] \vdash_0 \supset \sim \mathbf{typ}_8\,(\,\mathbf{m}_{9\,1}\,\mathbf{n}_{9\,1}\,\mathbf{x}_{9\,1}\,)$
$\vdash_2 \sim \mathbf{rel}\,[\,9\,9\,9\,6\,]\,\mathbf{Cl}\,(\,7\,)\,(\,\mathbf{m}_{9\,1}\,\mathbf{n}_{9\,1}\,\mathbf{x}_{9\,1}\,)$

(1) corresponds to the proper values of type-parameters and (2) to the other values.

Then we obtain, for example:

$$\models [10] \vdash_0 \vdash_2 \equiv \mathbf{rel}[9996]\,\mathbf{Cl}(7)\,(\overline{12}\,\overline{10}\,\mathbf{x}_{\overline{11}\,1})\,\mathbf{Class}[\overline{12}\,\overline{10}]\,\mathbf{x}_{\overline{11}\,1}$$

It should be noted that this is a theorem of $\mathbf{(2)_{\infty}}$ which contains the real variable $\bar{x}_{\overline{11}\,1}$ of $\mathbf{(0)_{\infty}}$. Such real variables are arbitrary constants of the system $\mathbf{(2)_{\infty}}$. Nevertheless to develop our theory further it is necessary to supplement them with the real variables of $\mathbf{(2)_{\infty}}$. For this purpose Hetper's postulate of generalized systems [1] will be employed.

We have [2]:

$$\models [10] \vdash_{0\,2} \equiv \mathrm{rel}[9996]\,\mathrm{Cl}(7)(\overline{12}\,\overline{10}\, x_{\overline{11}\,3})\,\mathrm{Class}[\overline{12}\,\overline{10}]\, x_{\overline{11}\,3}$$

The pattern $\vdash_{0\,2}\mathbf{E}$ has been employed to denote Hetper's generalized system.

I shall not discuss here the very interesting problems connected with the reconstruction of Cantor's theory. I wish simply to state that the analogy seems to be complete.[3] This may be verified by observing that we have invariable sequences of expressions, but that the sequences of classes which are contained in them are variable with regard to change of types. Consequently Zermelo's axiom cannot be proved for classes of classes, but it can be proved for classes of expressions.

On the other hand Zermelo's axiom can be proved for any classes in a finite system.[4] Thus we have infinite systems similar to those of classical mathematics. It should be noted that we can define Lebesgue's measure in these systems.[5] With the help of Hetper's generalized systems the arithmetic of real numbers is obtained with no restrictions. In this way a system of mathematics can be developed which is broader than classical mathematics and which contains neither the idealistic elements of Cantor's theory nor indiscutable propositions.

8. In both the system $\mathbf{[10]}$ and in the corresponding simple system of Hetper, it is possible to construct any system of symbolic logic. Here the systems of the pure and simplified theory of types will be constructed with the help of ordinary language. It should be noted that this construction differs from that of Russell but in practice the two are equivalent. An adequate construction would be much more complicated.

[1] Cf W. Hetper. "Simple Systems, Meta-systems and Generalized Meta-systems" (unpublished).
[2] Cf. *P. O. T. K.*
[3] *l.c.*
[4] *T L.*
[5] *M. L.*

I. PURE THEORY OF TYPES

(*a*) *Construction of Propositions and of Classes*

1. If $\vdash_0$ **(M2) Prop Class[KL] * H.I1** *is a theorem of the meta-system* **(M2)**, then $\vdash_0$**(M2)$_P$Prop Class[KL] * H.I1** *is a theorem of the system* **(M2)$_P$** *of the pure theory of types.*

2. If $\vdash_0$ **(M2)$_P$ Prop F** *is a theorem of* **(M2)$_P$**, and $\models$**[10] $\wedge \sim \equiv$ E.IK[1] gen[KL]$_1$ (E.ΠK) FG** *is a theorem,* then $\vdash_0$ **(M2)$_P$ G** *is a theorem of* **(M2)$_P$**.

3. If $\vdash_0$ **(M2)$_P$ Prop Π[K.0 L] XF** *is a theorem of* **(M2)$_P$**, and $\models$ **[10] (FX a$_{K.0L}$ G ([1]** *is a theorem,* then $\vdash_0$ **(M2)$_P$ Class[KL] Π[K.0L] a$_{K.0L}$ G** *is a theorem of* **(M2)$_P$**.

4. If $\vdash_0$ **(M2)$_P$ Class[KL] F** *is a theorem of* **(M2)$_P$**, then $\vdash_0$ **(M2)$_P$ Prop Class[KL] F** *is a theorem of* **(M2)$_P$**.

5. If $\vdash_0$ **(M2) Prop ε[KL] * G.I1 * H.I1** *is a theorem of* **(M2)**, then $\vdash_0$ **(M2)$_P$ Prop ε[KL] * G.I1 * H.I1** *is a theorem of* **(M2)$_P$**.

6. If $\vdash_0$**(M2)$_P$Prop ε[KL]XF** and $\vdash_0$**(M2)$_P$ Class[KL] G** *are theorems of* **(M2)$_P$**, then $\vdash_0$**(M2)$_P$Prop ε[KL]XG** *is a theorem of* **(M2)$_P$**.

7. If $\vdash_0$**(M2)$_P$Prop ε[.0KL]XF** and $\vdash_0$**(M2)$_P$Class[JK] H** *are theorems of* **(M2)$_P$** then $\vdash_0$**(M2)$_P$Prop ε[.0KL]HF** *is a theorem of* **(M2)$_P$**.

8. If $\vdash_0$ **(M2)$_P$ Prop E** and $\vdash_0$ **(M2)$_P$ Prop F** *are theorems of* **(M2)$_P$**, then $\vdash_0$ **(M2)$_P$ Prop /EF** *is a theorem of* **(M2)$_P$**.

(*b*) *Rules of Demonstration*

1. Nicod's syllogism.

2. Modus ponens.

3. The Principle of Generalization.

4. The Principle of deduction as applied to the propositions of **(M2)$_P$**.

5. THE PRINCIPLE OF LIMITED REDUCIBILITY

(*a*) If $\vdash_0 (\mathbf{M\,2})_{\mathrm{P}}\,\mathbf{Class}\,[\mathbf{K\,L}]\,\Pi[\mathbf{K\,.0\,L}]\,\mathbf{a}_{\mathbf{K\,.0\,L}}\,/\,\varepsilon[\mathbf{K\,.1\,L}]\,\mathbf{a}_{\mathbf{K\,.0\,L}}\,\mathbf{e}_{.2\,\mathbf{L}\,1}\,\varepsilon[\mathbf{K\,.1\,L}]\,\mathbf{a}_{\mathbf{K\,.0\,L}}\,\mathbf{f}_{.2\,\mathbf{L}\,1}$ *is a theorem of* $(\mathbf{M\,2})_{\mathrm{P}}$, then $\vdash_0 (\mathbf{M\,2})_{\mathrm{P}}\,\mathbf{Reduct}\,(\mathbf{K\,L\,L})\,/\,\varepsilon[\mathbf{K\,.1\,L}]\,\mathbf{a}_{\mathbf{K\,.0\,1}}\,\mathbf{e}_{.2\,\mathbf{L}\,1}\,\varepsilon[\mathbf{K\,.1\,L}]\,\mathbf{a}_{\mathbf{K\,.0\,L}}\,\mathbf{f}_{.2\,\mathbf{L}\,1}$ *is a theorem of* $(\mathbf{M\,2})_{\mathrm{P}}$.

(*b*) If $\models[\mathbf{1\,0}]\,\mathbf{gen}[\mathbf{K\,.1\,L}]_1(.\overline{\mathbf{25}}\,\mathbf{K\,.}\Pi\mathbf{K})\,\mathbf{F\,G}$ *is a theorem*, if $\vdash_0 (\mathbf{M2})_{\mathrm{P}}\mathbf{Reduct}(\mathbf{KLLF})$ and $\vdash_0 (\mathbf{M2})_{\mathrm{P}}\mathbf{Class}[\mathbf{KL}]\mathbf{a}_{\mathbf{K.0L}}\mathbf{G}$ *are theorems of* $(\mathbf{M\,2})_{\mathrm{P}}$, then $\vdash_0 (\mathbf{M\,2})_{\mathrm{P}}\,\mathbf{Reduct}\,(\mathbf{K\,L\,L})\,\mathbf{G}$ *is a theorem of* $(\mathbf{M\,2})_{\mathrm{P}}$.

6. THE PRINCIPLE OF TRANSFORMATION

If $\vdash_0 (\mathbf{M\,2})_{\mathrm{P}}\,\mathbf{Prop}\,\varepsilon[\mathbf{K\,L}]\,\mathbf{X}\,\Pi[\mathbf{K\,.0\,L}]\,\mathbf{a}_{\mathbf{K\,.0\,L}}\,\mathbf{G}$ *is a theorem of* $(\mathbf{M\,2})_{\mathrm{P}}$, if $\models[\mathbf{1\,0}]\,(\mathbf{G}\,\mathbf{a}_{\mathbf{K\,.0\,L}}\,\mathbf{X\,F})[\mathbf{1}]$ *is a theorem*, then $\vdash_0 (\mathbf{M\,2})_{\mathrm{P}} \equiv \mathbf{F}\,\varepsilon[\mathbf{K\,L}]\,\mathbf{X}\,\Pi[\mathbf{K\,.0\,L}]\,\mathbf{a}_{\mathbf{K\,.0\,L}}\,\mathbf{G}$ *is a theorem of* $(\mathbf{M\,2})_{\mathrm{P}}$.

Here $\mathbf{Reduct}\,(\mathbf{K\,N\,L})\,\mathbf{F}$ has been employed as an abbreviation of $\exists[.\mathbf{2\,N\,L}]\,\mathbf{g}_{.2\,\mathbf{N\,L}}\,\Pi[\mathbf{K\,L}]\,\mathbf{x}_{\mathbf{M\,L}} \equiv \varepsilon[\mathbf{K\,.1\,N}]\,\mathbf{x}_{\mathbf{K\,L}}\,\mathbf{g}_{.2\,\mathbf{N\,L}}$

$\varepsilon[\mathbf{K\,L}]\,\mathbf{x}_{\mathbf{K\,L}}\,\Pi[\mathbf{K\,.0\,L}]\,\mathbf{a}_{\mathbf{K\,.0\,L}}\,\mathbf{F}$.

All propositions of $(\mathbf{M\,2})_{\mathrm{P}}$ are propositions of $(\mathbf{M\,2})$, all theorems of $(\mathbf{M\,2})_{\mathrm{P}}$ would be theorems of $(\mathbf{M\,2})$, if the principle (*b*) of limited reducibility were not employed. The consistency of $(\mathbf{M\,2})_{\mathrm{P}}$ can then be proved very simply if it is supposed that $(\mathbf{M\,2})$ is consistent. If principle (*b*) of limited reducibility is employed Hetper's generalized meta-system $(\mathbf{M\,2})_{\mathrm{H}}$ is used rather than $(\mathbf{M\,2})$.

Then we have $(\mathbf{M\,2})_{\mathrm{H\,P}}$ instead of $(\mathbf{M\,2})_{\mathrm{P}}$ and the consistency of $(\mathbf{M\,2})_{\mathrm{H\,P}}$ follows from the simple remark that all the theorems of $(\mathbf{M\,2})_{\mathrm{H\,P}}$ are theorems of $(\mathbf{M\,2})_{\mathrm{H}}$. It is obvious that the consistency of $(\mathbf{M\,2})_{\mathrm{H}}$ must be assumed.

II. THE SIMPLIFIED THEORY OF TYPES

In order to obtain the system $(\mathbf{M\,2})_{\mathrm{S}}$ of the simplified theory of types, the rules of limited reducibility are replaced by the following rule of general reducibility:

If $\vdash_0 (\mathbf{M\,2})_{\mathrm{S}}\,\mathbf{Prop\,Reduct}\,(\mathbf{K\,N\,L})\,\mathbf{F}$ *is a theorem of* $(\mathbf{M\,2})_{\mathrm{S}}$, then $\vdash_0 (\mathbf{M\,2})_{\mathrm{S}}\,\mathbf{Reduct}\,(\mathbf{K\,N\,L})$ *is a theorem of* $(\mathbf{M\,2})_{\mathrm{S}}$.

It is obvious that in all the other rules $(\mathbf{M\,2})_{\mathrm{P}}$ is to be replaced by $(\mathbf{M\,2})_{\mathrm{S}}$.

The proof of the consistency of $(\mathbf{M\,2})_S$ cannot be obtained in a simple way because the principle of general reducibility is not compatible with $(\mathbf{M\,2})$.

However, the classical methods can be applied here without any difficulty. For example, Gödel's theorem can be proved in its primitive form and Church's theorem can also be proved. The latter theorem as applied to first-order classes of expressions states that there exists no effective method of deciding which propositions are provable.[1] It is a very important theorem since it enables us to see why the researches on this subject inaugurated by Löwenheim [2] and Skolem [3] and continued by the most eminent mathematicians could never be successful.

[1] Cf. B. Rosser: "An Informal Exposition of Proofs of Gödel's Theorem and Church's Theorem," *The Journal of Symbolic Logic*, vol. 4, 1939, p. 55.
[2] Leopold Löwenheim: "Über Möglichkeiten im Relativkalkül," *Mathematische Annalen*, Bd. 76, 1915, pp. 447–470.
[3] Th. Skolem: "Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit, etc," *Skrifter utgit av Videnskapsselskapet i Kristiania*, i, Matematisk-naturvidenskabelig klasse, 1920, no. 4, 36 pp.

Chapter VIII

THE FUNDAMENTAL CONCEPTS OF MATHEMATICAL ANALYSIS

1. It has been shown that the foundations of arithmetic were obscured by confused metaphysical discussion, and that the theory of classes developed from the confused idealism of Cantor. A similar statement may be made concerning classical mathematical analysis. It may be said that the very mathematicians who sought to remain within the domain of accurate reasoning made no progress and did not attain the desired end. On the other hand those who did not hesitate to trust their vision of reality triumphed decisively.

It has previously been remarked that in so far as the Greeks created euclidean geometry they cannot be regarded as finitists. However, the Eleatics were undoubtedly finitists and did everything in their power to check the development of the concept of the infinitesimal. This attitude characterized the arguments of the famous Zeno of Elea to whom some ascribe uncommon profundity and whom others deride unmercifully. It will now be seen that these arguments of Zeno were fundamentally false and can be justified only in terms of the confused concepts dominant in his day.

Zeno desired to prove that motion is impossible. With this in view he proposed the following arguments:

(1) "If everything is in rest or in motion in a space equal to itself and if what moves is always in the instant, the arrow in its flight is immovable." [1]

This argument is hopelessly confused.

According to Russell and Bergson, Zeno makes much of the point that the correspondence between the points of space and the moments of time does not cover the phenomenon of change which occurs during actual motion.

Russell disposes of this argument by rejecting the concept of change as having no clear meaning.[2] Bergson on the other hand makes this concept fundamental and therefore denies

[1] Bertrand Russell: *The Principles of Mathematics*, Cambridge, 1903, 2nd ed., New York, 1938, p. 350.
[2] Russell: *l.c.*, pp. 350–352.

any profound significance to an interpretation of movement with the help of an analysis in terms of points.[1]

Here it must be noted that an analysis in terms of points permits a certain number called the velocity, which in the case of rest is 0, to be assigned to every world point,[2] i.e. to every set of numbers which fixes the position of a moment of time and of the point of space which corresponds to it. Consequently it may be asserted that the attribution of a velocity to a world-point suffices to represent the fact that the state of a moving body undergoes change at that world-point.

It must be added here that according to contemporary physicists, the concept of velocity for a precisely determined position in space does not correspond to reality and becomes meaningless.[3] However, if it is desired that a purely sensationalistic position be maintained, no meaning can be attributed to the concept of a world-point and a *space* cannot be said to be *equal to itself*. This apparatus of concepts was borrowed from geometry, where alone it is possible to construct the concept of the velocity at a point, but neither is there any basis for the formulation of Zeno's argument.

(2) "There is no motion, for what moves must reach the middle of its course, before it reaches the end."[4]

In other words this means that motion is impossible for it cannot even begin.[5]

In this argument the fundamental supposition seems to be that it is possible to speak significantly of a temporal segment but not of a moment of time.

Actually there is no first temporal segment in which motion could begin, unless the division of temporal segments *ad infinitum* be permitted. If, however, such an operation is held to be permissible, the purely geometrical thesis, that moments of time correspond to points is being maintained.

If the concept of a moment is accepted, the answer to Zeno's question is simple. Motion begins at the moment a body is at the beginning of its path but at each subsequent moment

[1] Henri Bergson: *Time and Free Will*, translated by F. L. Pogson, London, 1910, pp. 112–117.

[2] Cf. 9.6.

[3] Cf. Erwin Schrödinger: "Indeterminism in Physics," translated by W. H. Johnston, *Science and the Human Temperament*, New York, 1935, p. 62.

[4] Russell: *l.c.*, p. 348.

[5] Cf. Jan Sleszyński: "O pierwszych stadjach rozwoju pojęć nieskończonościowych „*Poradnik dla Samouków* (" Concerning the First Stages of the Development of the Concept of the Infinitesimal," *Home Study Course*). Warszawa, 1923, tom iii, p. 56.

the body is not in motion. In other words motion begins at the earliest limiting moment of each of the temporal segments in which motion occurs.

Despite this explanation Sleszyński writes [1]:

> "The following fact is involved in the arguments of Zeno: There exists no point which lies next to a given point. Similarly there exists no moment of time which immediately follows a given moment. These facts make all change unintelligible. Thought is composed of a number of acts each of which requires a finite interval not less than a certain size. The number of these acts is therefore finite for everyone. Consequently it is impossible to know an infinite number of moments."

More confused reasoning can hardly be imagined. In the first place the motion observed must contain exactly as many elements as there are acts of thought. In the second place while it is true that an infinite number of moments cannot be conceived, this has nothing to do with the fact that the number of acts of thought is finite, since no one has ever counted his acts of thought and since it is impossible to do so.

It is difficult to maintain that the character of motion is understood but neither can it be maintained that what is meant by such questions as: What is a table? What is a cow?, is understood any better. Zeno's arguments become even more unintelligible if they are taken seriously. Because he took them seriously, Bergson fell into the hopeless confusions of irrational metaphysics.

(3) It is worth while to discuss still another of Zeno's paradoxes: that of Achilles and the tortoise. I will employ Richard's [2] formulation of this paradox.

Achilles runs ten times as fast as the tortoise. If the first distance is denoted by 1, when Achilles has traversed this distance, the distance between Achilles and the tortoise will be 0·1, since the tortoise will have traversed the latter distance. When Achilles will have traversed the segment 0.1, the tortoise will have covered the distance 0.01 and the distance between Achilles and the tortoise will be 0.01. At subsequent moments the distance between Achilles and the tortoise will be 0.001, 0.0001, 0.00001, etc. Thus the distance between them is always decreasing, but will never equal 0.

Richard calls this paradox crude and he is right. If Achilles requires the time t to cover the distance 1, he will require

[1] Sleszyński: *l.c.*, p. 57.

[2] Jules Richard: *Sur la philosophie des mathématiques*, Paris, 1903, p. 118.

the time $0.1\,t$ to cover the next segment, etc. Consequently the times

$$t,\ 1.1\,t,\ 1.11\,t,\ 1.111\,t,\ 1.1111\,t,\ 1.11111\,t, \ldots \text{etc.},$$

which are measured on a clock, correspond to the following distances between Achilles and the tortoise:

$$1,\ 0.1,\ 0.01,\ 0.001,\ 0.0001,\ 0.00001, \ldots \text{etc.}$$

Obviously the total time cannot exceed $\frac{10}{9}\,t$ and therefore the entire race not only does not last an infinite length of time but is actually very short. It is true that to enumerate separately the ever decreasing distances between Achilles and the tortoise would require an infinite length of time, but this fact leads to nothing of any great interest.

The distance between Achilles and the tortoise at the moment $\frac{10}{9}\,t$ will clearly be 0. If it were greater than 0, e.g. x, it would be possible to assign a natural number n so large that 10^n would be greater than $\frac{1}{x}$ and the fraction $\frac{1}{10^n}$ would be less than x. But it is possible to write the fraction $\frac{1}{10^n}$ in the form $0.00\ldots1$, where $n-1$ zeros follow the decimal point. To this distance would correspond the time $1.11\ldots1$ where there are n 1's after the decimal point. But this number is certainly less than $\frac{10}{9}$.

2. The explanation of the sophism of Zeno of Elea which has been given, reduces to the determination of the limits of the given sequences.

In the third paradox, the following sequences of numbers were involved:

$$1,\ 1.1,\ 1.11,\ 1.111,\ 1.1111, \ldots \text{etc.}$$
$$1,\ 0.1,\ 0.01,\ 0.001,\ 0.0001, \ldots \text{etc.}$$

The number $\frac{10}{9}$ is the upper limit of the first sequence. This means that the terms of this sequence are always less than $\frac{10}{9}$ and there is no positive number which is so small, that the difference between $\frac{10}{9}$ and a certain term of the sequence would not be less than this number.

0 is the lower limit of the second sequence. This means that the terms of this sequence are always greater than 0 and there is no positive number which is so small that a certain term of the sequence would not be less than this number.

Thus the determination of upper and lower limits is a new mathematical operation which is no worse than other mathematical operations and in general is not difficult.

For example, it can be seen at once that the natural numbers have no upper limit and that their lower limit is 1, since there is no positive number so small that the difference between the first term of this sequence (i.e. the number 1) and the number 1 was not less than this number.

Further the so-called sum of the geometric series

$$1 + q + q^2 + q^3 + q^4 + \ldots, \text{ where } 1 > q > 0,$$

which, as is known, is

$$\frac{1}{1-q},$$

is seen to be the upper limit of the sequence: $1,\ 1 + q,\ 1 + q + q^2 \ldots$

The concept of a limit is much simpler than that of a bound, but the two concepts do not differ essentially.

Anyone who understands that the determination of limits is a mathematical operation which does not depend upon approaching infinity and which can be performed in various ways, even by *guessing*, will have overcome the only real difficulty in the foundations of infinitesimal analysis. Everything else is an elaboration of this fundamental idea.

The theory of limits contains some interesting surprises. For example, the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

has no upper limit, i.e. the sum of this series is greater than any natural number, if a sufficient number of terms is considered. But the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

has an upper limit which can be ascertained without too much difficulty.

It might be supposed that such surprises made it difficult for the Greeks to discover the concept of a limit. It can rightfully be inferred from the mode of reasoning they employed, that an operation which can be performed in certain cases but not in others was not to their taste.

Archimedes actually knew how to determine the limits of series in special cases, but did not construct the general concept of a limit. He determined the upper limit of a geometric series and gave an example which involves the determination of the area of the segments of a parabola.[1]

Sleszyński writes:

> "We see therefore that Greek scientists could deal with infinity, but not, I say, in a clear way. The great mind of Archimedes saw the general method which clearly involved infinity, but he was fully aware of the need for scientific precision. He realized that this method is not logically established but is a heuristic method, whose results and applications must be verified in another way.[2]
>
> "Whether or not this was the case, it is a fact that modern scientists had to invent the infinitesimal calculus in their own way." [2]

This was accomplished with the help of generally confused and dubious images. Sleszyński quotes the following passage from Kepler [3]:

> "Archimedes uses an indirect proof which leads to the impossible. Various opinions have been advanced on this type of proof. It seems to me that its meaning may be derived from the following example. The circumference of the circle $B\,G$ has as many parts as points, i.e. an infinite number; each of them can be regarded as the base of a certain isosceles triangle whose legs are equal to $A\,B$, so that the area of the circle consists of an infinite number of triangles whose vertices intersect at the centre of the circle. Let us now straighten out the circumference of the circle $B\,G$. Let $B\,C$ be equal to this length and $A\,B$ be

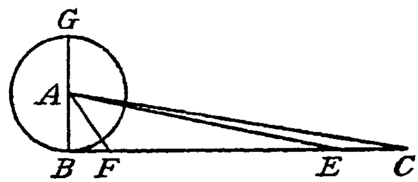

FIG. 6.

[1] Cf. Sleszyński: *l.c.*, p. 65. [2] Sleszyński: *l.c.*, p. 66.
[3] Sleszyński: *l.c.*, p. 68.

perpendicular to it. Let us now imagine that all the bases of these triangles or sectors are placed on the straight line *B C*. Let one such base be *B F* and then let a part which is equal to it, however small it may be, *C E*, be drawn. Furthermore let the points *F*, *E*, and *C* be joined to *A*. Because there are as many triangles *A B F*, *A E C* on the straight line *B C* as there are sectors in the area of the circle and because the bases *B F*, *E C* are equal to the bases of the latter, and because their common altitude is *B A*, the altitude of the sectors, the triangles *E A C* and *B A F* will be equal and each will equal one of the sectors of the circle. All the triangles having bases on the line *B C*, i.e. the triangle *B A C* which is composed of them all will be equal to the sum of all the sectors of the circle, i.e. the area of the circle is composed of all of them. The Archimedian method, which leads to the impossible is exemplified here."

This is an example of faulty reasoning which leads to a correct result. It suffices to observe, that Kepler speaks of triangles which have infinitely small bases and deals with them as he would deal with a finite quantity. He does not pay attention to the paradoxes which concern infinity. Nevertheless he obtains the correct result and becomes the great pioneer of the new mathematics.

This fact can be accounted for as follows: Kepler employs concepts which are not clear, but he does so within narrow limits and confines himself to the sphere within which they function correctly. He simply proceeds in the way in which one must proceed if the concepts cannot be stated as formuli. The reader will easily observe that this method is employed in the chapters of this book which deal with philosophical problems.

Cavalieri, Wallis, Pascal, Gregory St. Vincent, Fermat, Barrow, and others followed the footsteps of Kepler.[1]

Newton was undoubtedly the real creator of the infinitesimal calculus.

Leibniz introduced a convenient symbolism which determined the course of its further development.

3. A strange history which raises very disturbing questions is associated with the discovery of the infinitesimal calculus.

In 1666 Isaac Newton discovered the integral calculus. He collected his results in a work called *De Analysi per Aequationes Numero Terminorum Infinitas*, which, however, he did not publish. In 1673 Leibniz visited London, where he bought

[1] Cf Sleszyński: *l.c.*

on the recommendation of his friend Oldenburg a book entitled *Lectiones tum Opticae tum Geometricae*, which was written by Isaac Barrow, the teacher and collaborator of Newton. On his return he immediately began studies on the integral calculus. In 1675 he invented the integral sign $\int y\, dx$. This apparently purely formal result was actually the turning-point in the development of the infinitesimal calculus. In 1684 he published the results of his investigations, but referred to neither Newton nor Barrow. This fact caused unprecedented indignation in London. Leibniz alleged that he knew nothing of the work of Newton and that he had not employed that of Barrow. He did, however, use the letters *e* and *a* which had been employed by Barrow to denote infinitely small increments and he employed the expression *momentum* which had been invented by Newton, although the latter assigned a different meaning to the term. It is also certain that he had read Barrow's book because a copy of his notes has been preserved. Moritz Cantor sought to prove that the notes were made later because he did not want to suppose that Leibniz had lied. But it is plain that the character of a man like Leibniz warrants this supposition. Moritz Cantor wished to clear Leibniz of the charge of plagiarism at all cost, and therefore employed dialectical tricks.

Desiring to prove that while in London, Leibniz did not make the acquaintance of Collins, who was familiar with Newton's results, Cantor cited Oldenburg's letter to Leibniz in which Oldenburg informed Leibniz that he had transmitted his work to Collins. Cantor wrote :

> " Leibniz would have learnt this from Collins if he had met him, but their meeting without the knowledge of Oldenburg is inconceivable in the light of the close relations between the members of the London Society." [1]

All this is fine, but Oldenburg might have known of the meeting between Leibniz and Collins and that Collins informed Leibniz that he had received the latter's manuscript and therefore might have thought it suitable to let Collins know of this. Besides, in view of the relations of the time it would not be difficult to get possession of Collin's secret even without knowing him, if one were as skilful a diplomat as Leibniz.

To me the crucial facts are that after his stay in England

[1] Moritz Cantor : *Vorlesungen über Geschichte der Mathematik*, Leipzig, 1898, Bd. III, p. 28.

Leibniz began his studies on the infinitesimal calculus and that he did not acknowledge his familiarity with Barrow's text.

For this reason Cantor's arguments do not convince me and I am surprised that Sleszyński thought that they were decisive.[1] Moreover I am convinced by the following argument of Dühring :

> "Moreover it is not to be overlooked that the Newtonian method of fluxions originated in a way which was both natural and comprehensible to the investigators interested in inquiring into nature, while Leibniz's method of differentials when regarded from this point of view seems completely unmotivated and in the hands of its alleged discoverer remained without suitable applications to the system of nature." [2]

This entire dispute should be regarded as futile. Even if it were agreed that Leibniz dishonestly sought to deprive Newton of his invention, the only conclusion which would follow from this admission would be that Leibniz was not a man of great measure. But this conclusion is not new. Leibniz was a courtier who flattered powerful lords, and heroes are rarely found among such people. Moreover it is possible to be both a man of genius and a man without character. In any case while Leibniz sought to deprive Newton of his invention he did perfect it to such a degree that, as has already been stated, its future development was determined.

Bertrand Russell writes :

> "The English were misled by patriotism into adhering to his" (Newton's) "methods where they were inferior to those of Leibniz, with the result that after his death English mathematics was negligible for a hundred years." [3]

4. Familiarity with the concept of bound, the basic concept of the infinitesimal calculus, is not necessary to understand the essence of this calculus. It is quite sufficient to understand the concept of *limit*. The latter concept alone is necessary to define the velocity of a point at a given moment in all elementary cases of accelerated motion. Since the velocity is the relation between a distance and the time required to traverse that distance, velocities during very short periods of time can be discussed ; but it is meaningless to speak of

[1] Sleszyński : *l.c.*, pp. 53–4.
[2] Eugen Dühring : *Kritische Geschichte der Philosophie*, Berlin, 1869, p. 331.
[3] Bertrand Russell : *The Scientific Outlook*, New York, 1931, p. 39.

the velocity of a point at a given moment. If, however, the motion is such that the velocity invariably decreases or increases with the time, the velocity at a given moment can be regarded as the limit of all the velocities during the period from the given moment to some following moment.

If the motion of a freely falling point is considered, the relation between the time of descent, which is represented by t and the ordinate x, is $x = -5t^2 + c$, where c is the ordinate of the point at the moment represented by 0 and -10 is the value in round numbers of the acceleration. This equation leads to a figure which is a parabola intersecting the x-axis at the ordinate c and the t-axis at a point which represents the moment $\sqrt{\frac{c}{5}}$.

If the moment represented by t and a subsequent moment represented by $t + h$ are considered, the ordinates : $-5t^2 + c$ and $-5(t + h)^2 + c$ correspond to the abscissas which represent these moments. The number which represents the distance traversed by the point during the interval denoted by h is the negative of the difference between these abscissas, i.e. $10t \cdot h + 5h^2$. The negative of the number which is the ratio of the number measuring this distance and the number representing the interval of time h, i.e. $-(10t + 5h)$ denotes the mean velocity of the point during the interval between the moments represented by t and $t + h$. It obviously depends upon the number representing the length of the interval h and therefore decreases in absolute value as this number decreases. Obviously this formula cannot be employed in computing the velocity of the point at the moment represented by t since in this case no interval occurs. However, it can be said that *the velocity of the point at the moment represented by t is the upper limit of the number which represents the mean velocity computed from the moment represented by t to the moment represented by $t + h$.* This limit is $-10t$, i.e. the number which denotes the velocity of the point at the moment represented by t, is $-10t$.

It should also be noted that a negative rather than a positive increment might have been employed. Analogous reasoning would then be carried through until the end, when the lower and not the upper limit of the number which denotes the mean velocity would have been taken. The result would again have been $-10t$.

The value of this device lies in the fact that at the end the velocity of a point at a definite moment can be obtained.

At 0, 1, 2, 3, . . . seconds the point has the velocities 0, -10, -20, -30, . . . meters per second respectively. There is no need to refer to the number which denotes the mean velocity during a certain period of time.

The concept of the velocity at a point is not at all complicated. It proves to be a natural concept and what is more important it has little more in common with the infinitesimal than the concept of the least odd or perfect number.

Hence it follows that in a strict sense there was no reason to construct the fiction that some non-terminating sequence of approximations must be employed in solving this problem. In plain terms Zeno of Elea obstructed progress and his suggestion hampered even those who created the infinitesimal calculus.

As long as the motion under consideration is of such a kind that the mean velocity computed from the moment investigated to some subsequent moment or from some prior moment to the investigated moment invariably decreases or increases during some interval, no matter how short, only the concept of a limit is necessary to calculate the velocity. Moreover other types of motion are encountered only in dealing with cases which do not involve elementary problems. Not until then does the concept of *bound* prove necessary. This concept is not difficult, but to understand the traditional construction a certain skill in manipulating inequalities is necessary. I will give a simple construction of this concept below. Lectures on the infinitesimal calculus usually start with the concept of bound, and the concept of limit is introduced later. This is a wholly unnecessary complication of the subject.

The first attempt to formulate precisely the concept of bound was made by D'Alembert.

In the first half of the nineteenth century Cauchy gave the first entirely accurate definition of this concept.

Newton and Leibniz defined the velocity at a point as the ratio between an infinitely small increment of the distance to an infinitely small increment of the time. Leibniz denoted these increments by the symbols dx and dt; the velocity at a point was denoted by the ratio $\frac{dx}{dt}$.

The calculus was conceived somewhat as follows: The increment of the distance denoted by dx is the difference between the ordinates which corresponds to the increment of the time denoted by dt. Consequently at the moment t,

$$dx = -(10\,t + 5\,dt)\,dt$$

and the ratio $\frac{dx}{dt}$ satisfies the equation:

$$\frac{dx}{dt} = -(10\,t + 5\,dt).$$

The symbol $\frac{dx}{dt}$ was said to represent the *derivative of x with respect to the variable t.*

This notation is employed even to-day and it is said that the number which represents the velocity at the moment denoted by t is $\frac{dx}{dt}$ and that $\frac{dx}{dt} = -10\,t$. However, this is no longer regarded as the ratio between two increments but simply as the upper limit of numbers which can be obtained from the expression $-(10\,t + 5\,h)$ by substituting an arbitrary positive number for h.

It is therefore clear that the concept of infinitely small increments was not indispensable.

5. The definition of the concept of a function of a real variable which is given in classical texts raises serious questions. In Goursat's *Cours d'analyse mathématique*, the following passage may be found.

> "When two variable quantities are so related that the value of one of them depends on the value of the other, they are said to be functions of each other." [1]

The reader may think that concrete quantities, for example, a moment and the number which is thrown at that moment at dice-playing or a number just thought of, are functions of each other. These quantities are in fact related in such a way that the value of one depends upon the other.

Examples of this kind are often adduced to explain the concept of functions. Consequently the meaning of this concept has been completedly obscured.

[1] É. Goursat: *A Course in Mathematical Analysis*, translated by Earle R. Hedrick, Boston, 1904, vol. i, p. 2.

The following much more accurate definition is found in Tannery's *Introduction à la théorie des fonctions d'une variable:*

> "A function y of x is defined in the interval (a, b) if to each value of x, belonging to this interval corresponds a determinate value of y." [1]

But this definition also leaves much to be desired. The *value* for x or y is unknown because x and y are letters which have no characteristic which can be called a value. The meaning of the word *corresponds* is also unknown.

This information cannot be obtained until specific examples are examined. If, for example, it is said that a function of the letter x is defined in the interval (a, b), whenever the value 0 corresponds to any rational number which is contained in this interval and whenever the value 1 corresponds to any irrational value, some idea of the meaning of this definition is obtained. If it is said that $\sin x$ is a function of the variable x, since a certain *value* of $\sin x$ corresponds to every *value of* x, the definition seems much clearer. When finally it is realized that every expression which can be constructed with the help of the signs of the given system and which contains a variable letter, is a function of that letter, the feeling arises that the meaning of a function is understood.

But then the following question may be raised:

Why did not the author simply say that every expression which can be constructed with the help of mathematical signs and which contains the letter x, is a function of that letter in the interval (a, b), if this expression becomes a number whenever any number greater than a and less than b is substituted for x?

Would it not be preferable to say that x is a letter and that what is intended is not a metaphysical correspondence but a correspondence between an expression which contains the letter x and the expression which is obtained from it by substituting a certain number for x?

The very use of the expression $(x + 1)$ involves the concept of a mathematical function. The great mistake of teaching in the past and even at the present time (although recently school syllabi show definite improvements in this regard) is that this fact was not taken into consideration even in the most elementary studies. As a result the student imagines

[1] Jules Tannery: *Introduction à la théorie des fonctions d'une variable*, Paris, 1886, p. 99

that letters and the expressions which can be constructed from them are a new kind of general number. Later he must combat with great effort and frequently without success the prejudice which arose in this way. It must be recognized explicitly that $(x + 1)$ is not a new number, but an expression which becomes a number when and only when some number is substituted for x. This means that this expression will not become a number until some number is *substituted* in it for the letter x.

It should be observed that the function $(x + 1)$ is determined in all possible intervals, because any number can be substituted for x. The function $\sqrt{x + 1}$ is governed by the condition $x \geqslant -1$, and therefore is determined in all intervals (a, b) whose lower limit satisfies the condition $a \geqslant 1$.

The conditions which must be satisfied by a number so that in a given function it may be substituted for x will be said to fix the domain of this function.

Classical mathematicians do not simply say that functions are expressions because they have not sufficiently well developed methods of construction. The methods at their disposal do not even suffice to construct as simple a function as the function whose value is 0 at the point 0, and 1 at all other points.

Such functions can be described but not constructed in the theory of relations. They can, however, be constructed if functions are regarded as expressions. To demonstrate this fact the function just mentioned will be constructed. Signs of type will be neglected because it would be pedantic to employ them where no misunderstandings can occur.

The construction of the propositions: **Real E**, $\overset{=}{\mathbf{A}}\ \mathbf{E}\ \mathbf{1}$, $\overset{=}{\mathrm{Cl}}\ \mathbf{F}\ \mathbf{\Lambda}$, presents no difficulty.

The first proposition may be read: **E** *is a real number*, and the others: **E** *is a rational number arithmetically equal to the number* **1**, **F** *is the null class*. By means of these propositions can be constructed the proposition:

$$\wedge\ \mathbf{Real}\ \mathbf{E}\ \wedge \sim \overset{=}{\mathrm{Cl}}\ \mathbf{E}\ \mathbf{\Lambda}\ \overset{=}{\mathbf{A}}\ \mathbf{F}\ \mathbf{1},$$

which is abbreviated by the symbol $\Phi\,(\mathbf{E}\,\mathbf{F})$.

The expression $\mathbf{\Pi}\, y\, \Phi\,(x\, y)$ is the desired function. It is the class of all rational numbers which are equal to **1**, when the real number x is not the null class and therefore differs from **0**. When x is zero, the class is the null class and therefore equals **0**.[1]

Attention will now be directed toward a problem which has long been known but was not settled or even formulated precisely by mathematical students until the present day. In classical mathematics relations between functions were not clearly distinguished from relations between their values.

Frequently in texts on analysis, the function $y = f(x)$ rather than the function $f(x)$ is discussed. It is difficult to imagine a more ambiguous way of talking. In the first place $y = f(x)$ is an equation or, more correctly, a propositional function, but in no case is it a mathematical function. In the second place, when it has been realized that $f(x)$ is a function, it cannot be said that the equation $y = f(x)$ is a relation between two functions.[2] The letter y is a function of the variable y, $f(x)$ is a function of the variable x. These two functions are therefore fundamentally different. The equation in question is simply a relation between two unknowns x and y, and that is all which may be said. The very fact that the domain of the variable in the equation can differ from the domain of the variable in the two functions, shows that the equation is not a relation between functions. For example, the domain of the function $\frac{x}{x}$ contains all numbers with the exception of 0; but the domain of the function $\frac{x-1}{x-1}$ contains all numbers with the exception of 1, and the domain of the equation $\frac{x}{x} = \frac{x-1}{x-1}$ contains all numbers with the exception of 0 and 1.[3]

It is frequently said that functions are equal throughout the interval (a, b) if they are defined throughout this interval and if they have the same values at the same points. It therefore follows, that the functions $\sqrt[3]{x^2}$ and $(\sqrt[3]{x})^2$ are equal throughout every interval. Yet in the first case the cube root of x^2 is being considered and in the second the square of $\sqrt[3]{x}$. This state of affairs requires that it be shown that the operations introduced do not depend upon the structure of the functions, but upon their values alone. Only too often

[1] Examples of other such constructions may be found in *M M.A*, p 258.

[2] Cf. Gottlob Frege: *Grundgesetze der Arithmetik*, Bd. II, Jena, 1903, p. 148.

[3] Cf the interesting note in Gutkowski and Wilkosz: *Algebra elementarna* (*Elementary Algebra*), Warszawa, 1925 (5th edition), pp 125–6.

classical mathematicians fail to take account of the fact that in investigating this problem they are not working in the field of mathematics but in that of semantics. At all events they do not mention this fact to their pupils, who therefore remain perplexed.

If certain wide fields are excluded from the domain of functions, reasoning is governed by a theory of types. For example, when a function of a real variable $\frac{1}{x}$ is distinguished from a function of a natural variable $\frac{1}{i}$ (i.e. a variable which takes only integers as values), it is clear that the distinction drawn is based upon a theory of types. Under these conditions the apparently greater clarity of the classical formuli in comparison with the semantical formuli is a result of the abandonment of precise notation. However, it must be kept in mind that once the symbolic apparatus of semantics has been worked out, a simplified symbolism can be introduced with no fear of misunderstandings.

Signs of type can be neglected where there is no fear of misunderstandings. Semantical symbolism thus becomes clearer and simpler than classical symbolism.

6. The weakness of classical symbolism is revealed in the discussion of sequences and series. To this day a sequence is represented by the description :

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

It should, rather, be explicitly said that a sequence is a relation between the values of a function of a natural variable n. What is even worse is that the functional pattern a_n is used simultaneously with the pattern $f(n)$ and it is not stated that both patterns serve identical ends. Moreover it should be observed, that if a_1, a_2, a_3, . . . are given numbers, they could not at the same time represent an infinite sequence, since it is possible neither to write down nor think of infinitely many numbers. In order that it might be possible to speak of a sequence or progression of numbers, a function of a natural variable a_n or $f(n)$ must be given, for only then can it be said that the nth term of the sequence is a_n or $f(n)$.

This confusion of concepts appears in especially troublesome form in connection with series.

The symbol:

$$a_1 + a_2 + a_3 + \ldots + a_n \ldots$$

is still in continual use despite the fact that it suggests the completely false idea that it is possible to add *ad infinitum*. To this symbol is subjoined the sign $\sum_{k=1}^{n} a_k$, which denotes the sum of the first n terms of the series. Thereby is emphasized the idea that it is possible to deal with other sums in addition to such as have been explicitly indicated.

Actually the study of series reduces to the investigation of expressions of the form $\sum_{k=1}^{n} a_k$ or $\sum_{k=1}^{n} f(k)$ or also $\sum_{k=1}^{n} f(k, x)$, etc.

These expressions are simply functions of the integral variable n and in some cases of other variables. The letter $\dot{k}$ is an apparent variable of a type for which nothing can be substituted.

It must still be explicitly stated that these functions cannot be formulated with the help of the usual concept of a sum. In constructing such functions it is absolutely necessary to employ such intervals as Dr. Hetper [1] used in defining the operations of multiplication and involution.

With the introduction of the patterns a_n, $f(k)$, $f(x)$ it is clear that the investigation has been carried into the field of semantics, because such patterns as these are neither numbers nor functions, but rather certain particular expressions which become functions if suitable substitutions are made. Students tend to treat the pattern $f(x)$ in the same way as they treat the function $\sin x$. I must confess that this gross error is not entirely their fault.

The patterns which have been mentioned are constructed in such a way that they can be employed in performing a series of operations which cannot be carried out within pure mathematics. In particular it is possible to construct from the old patterns such new ones as $f(x + h)$, $f(f(x))$. These operations demand exactness and precision. For example, it might happen that $f(i)$ is not a whole number. It would then be

[1] Cf. W. Hetper: "Semantische Arithmetik," *Sprawozdania Towarzystwa Naukowego Warszawskiego*, tom XXVII, 1934, Wydział III, p. 17 f.

necessary to keep in mind that $f(f(i))$ has no meaning. It must also be kept in mind that in an expression in which $f(0)$ appears $\frac{1}{x}$ cannot be substituted for $f(x)$; if, however, one confines oneself to real numbers the substitution of $\sqrt{x-1}$ is not permissible. It is evident that a large apparatus of tacitly accepted conventions is involved here. I think that silence on this point gives many mathematicians and virtually all theoretical physicists the impression that mathematics is an occult science.

It will suffice for our present purposes to construct the class:

$$\mathbf{\Pi\,[\,M\,L\,]\;a_{M\,L}\wedge Rat_{M}\,a_{M\,L}\wedge Real\,[\,M\,.1\,L\,]\;X}$$
$$\mathbf{\wedge\;\exists\,[\,.4\,L\,.0\,L\,]\;y_{.4\,L\,.0\,L}\wedge Real\,[\,M\,.1\,L\,]\;y_{.4\,L\,.0\,L}}$$
$$\mathbf{\wedge\;\varepsilon\,[\,M\,.1\,L\,]\;a_{M\,L}\,y_{.4\,L\,.0\,L}\,rel\,[\,.4\,L\,.4\,L\,.0\,L\,]\;F\,(X\,y_{.4\,L\,.0\,L})},$$

which can be abbreviated by $\mathbf{F_{M\,L}\,(\,X\,)}$.

It is obvious that $\mathbf{F_{M\,L}\,(\,X\,)}$ is the same symbol as $f(x)$ except that in the former the types are indicated.

$\mathbf{Real\,[\,M\,.1\,L\,]\;X}$ has been employed here rather than just $\mathbf{Real\;X}$ since the types are not to be omitted. The symbol $\mathbf{Rat_{M}\,E}$ has been introduced to denote rational numbers of the type $\mathbf{M}$.

It is clear that $\mathbf{F_{M\,L}\,(\,X\,)}$ is the real number whose elements are rational numbers contained in a real number which has the relation $\mathbf{F}$ to $\mathbf{X}$.

An analogous construction can be made in the system of Whitehead and Russell, but not in the pure theory of types since it is impossible to deal with variable types in the latter theory.

It should be observed that the semantical theory of functions has not been considered here. I hope, however, to develop it fully in the near future.

7. The elimination of the differential from mathematical analysis by Cauchy made the construction of a system of theoretical physics much more difficult. For this reason mathematicians working in the field of analysis have never abandoned the concept of the differential but introduced it by means of the concept of a function which approaches 0 at the point 0. Until now the theory of differentials has never been precisely worked out. In courses on analysis only rather

general remarks concerning differentials can be found. This state of affairs seems to confirm the old prejudice that there exists an abyss between algebra and analysis which cannot be bridged.

To remedy this evil it is absolutely necessary to show that the concept of number can be generalized in such a way as to include the concept of a differential by employing only the concepts upon which algebra is based. The apparatus of concepts of rational semantics permits the complete solution of this problem.

Instead of speaking of functions of a real variable which approaches 0 at the point 0, it is possible to speak of arbitrary sequences of real numbers. Sequences which invariably take on the same value at points sufficiently far from 1, i.e. sequences which beginning with a certain term are equal, can be assembled in separate classes which are called sequential numbers.[1]

Nevertheless in conformity with a remark of Herzberg, sequences will be dealt with directly.

The following definitions are posited :

1. If E is a real number, E is an 0-order number.

2. If E is an n-order number, for any value of a natural variable I, then the sequence $\bar{I}\,E$ of the values of E is an $(n + 1)$-order number.

If a, a_i, a_{ik} and a_{ikl} are functions of the natural variables i, k and l, the following are numbers :

$$\bar{\imath}\,a,\ \bar{\imath}\,a_i,\ \bar{k}\,\bar{\imath}\,a_i,\ \bar{\imath}\,\bar{k}\,a_i,\ \hat{l}\,\bar{k}\,a_i,\ \bar{l}\,\bar{k}\,a_{lk},\ \bar{l}\,\bar{k}\,\bar{\imath}\,a_{lki}\ldots$$

The following are likewise numbers :

Order	Number	The corresponding sequence
1	$\bar{\imath}\,0$	0, 0, 0, . . .
1	$\bar{\imath}\,i$	1, 2, 3, . . .
2	$\bar{\imath}\,\bar{k}\,0$	$\bar{k}\,0,\ \bar{k}\,0,\ \bar{k}\,0$
2	$\bar{\imath}\,\bar{k}\,k$	$\bar{k}\,k,\ \bar{k}\,k,\ \bar{k}\,k$
2	$\bar{\imath}\,\bar{k}\,i$	$\bar{k}\,1,\ \bar{k}\,2,\ \bar{k}\,3$
2	$\bar{\imath}\,\bar{k}\,\frac{i}{\bar{k}}$	$\bar{k}\,\frac{1}{\bar{k}},\ \bar{k}\,\frac{2}{\bar{k}},\ \bar{k}\,\frac{3}{\bar{k}}$

Numbers which contain an infinity of members all of which are equal to 0 will be called null numbers.

Convention : In these constructions of sequential numbers

[1] Cf. *U. H. M*, p 463 ff

the number 0 will be taken for any expression $\frac{a}{0}$ unless there is an infinity of such expressions. In the latter case the corresponding pattern will not be used at all.

For example $\bar{\imath}\,\frac{1}{i-1}$ is the sequence 0, 1, ½ . . .

$$\bar{\imath}\,\frac{1}{\sin\frac{\pi}{2}i} \text{ is a meaningless expression.}$$

The nth member of the sequence X will be denoted by $[n]\,X$. For example :

$$[n]\quad \bar{\imath}\,a = a$$
$$[n]\quad \bar{\imath}\,a_\iota = a_n$$
$$[n]\quad \bar{k}\,\bar{\imath}\,a_{\iota k} = \bar{k}\,a_{nk}$$

Note that the symbol $[N]\,E$ should not be employed if E contains N.

Small latin letters will be employed to denote the real numbers and large latin letters to denote sequences.

THE DEFINITION OF EQUALITY

I. If $[n]\,X = a$ for almost all values of n, i.e. except for a finite number of values of n,

$$X = a$$

II. If $[n]\,X = [n]\,Y$ for all values of n then

$$X = Y$$

Examples :

$$a = \bar{\imath}_1\ a = \bar{\imath}_2\bar{\imath}_1\,a = \bar{\imath}_3\bar{\imath}_2\bar{\imath}_1\,a\ldots$$

$$\bar{n}\,\bar{m}\,\frac{m-1}{m(m-1)} = \bar{n}\,\bar{m}\,\frac{1}{m}$$

DEFINITION OF GREATER AND LESS

I. If $[n]\,A > a$ for almost all values of n, then :

$$A > a$$

II. If $a > [n]\,X$ for almost all values of n, then:

$$a > X$$

III. If $[n]\,X > [n]\,Y$ for almost all values of n, then :

$$X > Y$$

DEFINITION OF FUNDAMENTAL OPERATIONS

If $X' = X$ and $Y' = Y$, where X' and Y' are of the same order, then

$$X + Y = \bar{n}\,(\,[n]\,X' + [n]\,Y'\,)$$
$$X - Y = \bar{n}\,(\,[n]\,X' - [n]\,Y'\,)$$
$$X \cdot Y = \bar{n}\,(\,[n]\,X' \cdot [n]\,Y'\,)$$
$$\frac{X}{Y} = \bar{n}\,\frac{[n]\,X'}{[n]\,Y'}$$

where Y is not a null number.

For example

$$\bar{\imath}\,i + a = \bar{n}\,(\,[n]\,\bar{\imath}\,i + [n]\,\bar{\imath}\,a\,) = \bar{n}\,[n + a]$$
$$\bar{\imath}\,i \cdot \bar{\imath}\,(i - 1) = \bar{n}\,n\,(n - 1)$$

DEFINITION OF POSITIVE AND NEGATIVE NUMBERS

Positive numbers are greater than 0 and negative numbers less than 0.

DEFINITION OF DIFFERENTIALS

Positive numbers which are less than any positive $(n - 1)$-order number are n-order differentials.

The n-order differentials are denoted by $d_n\,x,\ d_n\,y,\ d_n\,z\ \ldots$

For example $\bar{\imath}\,\frac{1}{i}$ is a 1-order differential

$\bar{k}\,\bar{\imath}\,\frac{1}{i}$ is a 2-order differential

$\bar{l}\,\bar{k}\,\bar{\imath}\,\frac{1}{i}$ is a 3-order differential

DEFINITION OF A LIMIT-VALUE OF A SEQUENTIAL NUMBER

The n-order number G is an n-order limit-value of A if for the differential $d_{n+1}x$

$$-\,d_{n+1}x \leqslant A - G \leqslant d_{n+1}x$$

Then we write :

$$\lim\,(n)\,A = G$$

It can be proved that there are never two different n-order limit-values of a given number.

We have, e.g. $\lim\,(0)\, d_n\, x = 0$

$$\lim\,(n+p)\, d_n\, x = d_n\, x$$

$$\lim\,(0)\, a = a$$

$$\lim\,(0)\, \bar{i}\, \frac{(x+\frac{1}{i})^2 - x}{\frac{1}{i}} = 2\,x$$

DEFINITION OF n-ORDER FUNCTIONS

The expression $f(X)$ is an n-order function of X in the domain Ω, if for any n-order value of $\Omega, f(X)$ is a uniquely determined n-order number.

If the recursive pattern

$$f(X) = n\, f\,([n]\, X)$$

is employed, the following n-order elementary functions are obtained :

$$\text{(E)} \qquad \begin{aligned} -X &= \bar{n}\,(-[n]\,X) \\ |X| &= \bar{n}\,|[n]\,X| \\ e^x &= \bar{n}\, e^{[n]\,x} \\ \log X &= \bar{n} \log\,[n]\,X \\ \sin X &= \bar{n} \sin\,[n]\,X \\ \arcsin X &= \bar{n} \arcsin\,[n]\,X \end{aligned}$$

Also :

$$E(X) = \bar{n}\, E\,([n]\, X),$$

where $E(x)$ is the next integer not greater than x.

We have the following definitions for series

$$\sum_{k=1}^{E(X)} f(k, Z) = \bar{n} \sum_{k=1}^{E(\,[n]\,X)} [n]\, f(k, Z)$$

LIMITS

The following definitions are posited for n-order functions.

$$\lim_{Z \leftarrow X} f(X) = \lim\,(n)\, f(Z + d_{n+1}x)$$

$$\lim_{X \rightarrow Z} f(X) = \lim\,(n)\, f(Z - d_{n+1}x)$$

$$\lim_{X \doteq Z} f(X) = \lim_{Z \leftarrow X} f(X) = \lim_{X \rightarrow Z} f(X)$$

$$\lim_{X \doteq \infty} f(X) = \lim\,(n)\, f\left(\frac{1}{d_{n+1}x}\right)$$

$$\lim_{X \doteq -\infty} f(X) = \lim\,(n)\, f\left(\frac{-1}{d_{n+1}x}\right)$$

for any $d_{n+1}x$.

The derivative and the definite integral are then defined as follows:

$$f^1(X) = \lim_{Z \doteq X} \frac{f(Z) - f(X)}{Z - X}$$

$$\int_A^B f(X)\, dX = \lim_{X \doteq \infty} \quad (n)E(B - A)\,(X) \sum_{k=1} f\left(A + \frac{k}{X}\right)\frac{1}{X}$$

Thus the definite integral is the limit of an infinite series of infinitely narrow rectangles.

MULTIPLE SEQUENTIAL NUMBERS

A simple analysis of multiple series and multiple integrals cannot be given in terms of simple sequential numbers. For this purpose *multiple-sequential numbers* which are double, triple, etc., sequences are employed.

THE FULFILLED LINEAR CONTINUUM OF 1-ORDER NUMBERS

We will begin with the real 1-order numbers $\bar{\imath}\, a$, and $\bar{\imath}\, i$; the fundamental operations and the elementary functions (E) will be assumed, where the domain of X in $\sin X$ is limited to the real numbers.

In this way a set of numbers is obtained which is called the elementary continuum of 1-order numbers. Step by step this continuum can be completed by constructing new numbers. With the help of Zermelo's axiom it can be proved that there is a *fulfilled continuum*, i.e. a linear field which cannot be supplemented by a larger linear field.

It should be noted that there is no way of defining a norm of sequential numbers. This conclusion is derived from a general theorem of the theory of functional operations of Banach,[1] which was proved by Mazur.[2]

NORMAL 1-ORDER FUNCTIONS

The 1-order function $f(X)$ will be called a *normal function* if

$$f(X) = \tilde{n}\, f_n\,([n]\, X)$$

where $f_n\,(X)$ is a function of a real variable.

The theorems

$$f'(X) = n\, f_n'\,([n]\, X)$$

$$\int_A^B f(X)\, dX = n \int_{[n]A}^{[n]B} f_n\,(x)\, dx$$

can be proved without any difficulty.

The theorem:

$$\int_0^\infty f(X)\, dX = \tilde{n} \int_0^\infty f_n\,(x)\, dx$$

can also be proved.

DIRAC'S FUNCTIONS

The 1-order function

$$\frac{1}{\pi}\,\frac{d_1\, a}{X^2 + (d_1\, a)^2}$$

denoted by $\delta\,(X)$ is clearly a normal function. The ordinary calculus of classic analysis can therefore be applied to this function. It will be shown that it is a Dirac's function.

If X is a member of a fulfilled linear continuum, the corresponding value of $\delta\,(X)$ is also a member of this continuum and $\delta\,(X)$ has the following properties:

X	(X)
0	$\frac{1}{d_1\, a}$
$n\sqrt{d_1\, a}$	$\frac{1}{n^2 + d_1\, a}$
$(d_1\, a)^{\frac{n-1}{2n}}$	$d_1\, y$
∞	0

[1] S Banach: *Théorie des opérations linéaires,* Warszawa-Lwów, 1932.

[2] S. Mazur: "Sur les anneaux linéaires," *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, Paris, Tome 207, 1938, pp. 1025–1027.

The value of $\delta(X)$ at the point 0 is infinite. It decreases to a real number in the infinitely small interval $(0, n\sqrt{d_1 a})$.

In the infinitely small interval $\left(n\sqrt{d_1 a},\ (d_1 a)^{\frac{n-1}{2n}}\right)$ it decreases to a differential and it is constantly decreasing along the axis $\left((d_1 a)^{\frac{n-1}{2n}}, \infty\right)$, its limit being 0.

For negative values of X the same values are obtained for $\delta(X)$.

Therefore :

$$\delta'(X) = -\frac{1}{\pi}\frac{2\,X\,d_1 a}{\left(X^2 + (d_1 a)^2\right)^2}$$

and $\delta^{(n)}(X)$ is obtained by the elementary calculus.

By applying this calculus the following equalities may also be obtained :

$$\int_0^A \delta(X)\,dX = \frac{1}{\pi}\arctan\frac{A}{d_1 a}$$

$$\int_0^{\sqrt{d_1 a}} \delta(X)\,dX = \frac{1}{2} - \frac{1}{\pi}\arctan\sqrt{d_1 a}$$

$$\int_0^{\infty} \delta(X)\,dX = \frac{1}{2}$$

$$\int_{-\infty}^{\infty} \delta(X)\,dX = 1$$

Chapter IX

PROBLEMS OF THE METHODOLOGY OF THE EXACT SCIENCES

1. The system of rational metamathematics can be regarded as a tool quite similar to a counting machine. The former enables us to obtain results in addition to those obtained by a counting machine. In particular this system can be utilized to obtain expressions which permit the prediction of other expressions at a later stage. While a counting machine permits the prediction of the results of definite experiences, namely of the results of counting, the scope of the experience which can be apprehended by the system of rational metamathematics cannot be established in advance.

The ability to formulate in terms of a pattern experiences which depend upon making spatial measurements was the first advance over a counting machine. In other words the system of rational metamathematics includes the science called geometry.

Geometry is an experimental science. It depends upon the measurement of segments, angles, and areas. The Egyptians conceived it in this way and it has remained essentially the same up to this very day. To-day what is generally regarded as geometry, i.e. what is included in textbooks, is the peculiar mixture of experimental geometry and the geometrical metaphysics which was inherited from the Greeks as Euclid's *Elements*.

Closer consideration of the constructions with which euclidean geometry operates reveals that they are as inaccessible to the imagination and to experience as Cantor's aggregates.[1] The illustrations taken from experience, which are given in textbooks, whether of straight lines, planes, or points differ fundamentally from what is actually meant by these terms. Their only value is that they create the illusion that they explain something. As a result every student is convinced that he knows what a point, straight line, or plane is. He speaks of them with great freedom just as he speaks of grains of sand, the rays of the sun, or the surface of a calm lake. He does not realize that the common properties of both these types of objects are few in comparison with their differences. Once

[1] Cf. 6.5.

this is taken into account it will be difficult not to admit that in the last resort we have no idea what the constructions of the geometry of Euclid are. It must therefore be concluded that the latter operates with fictional objects and consequently largely with words which have no meaning.

The idealistic belief of the Greeks in the existence of points, straight lines, and planes led them to create a science which later became one of the most powerful arguments in the struggle with the problem of reality and which quite unexpectedly permitted the extension of the bounds of experience. This science was the first example of a well-worked-out conceptual apparatus. These facts have been powerful arguments for the metaphysical character of the foundations of science.

Unquestionably it was on the basis of this conviction that Spinoza attempted to construct the foundations of metaphysics by employing the methods of geometry. The failure of this attempt became an incentive for intensive researches on the essence of geometry. Although the analogy between Greek geometry and metaphysics is very striking, what is of importance here is the criteria by means of which they are to be distinguished. The former is characterized by its universal possibilities of development, the unshaken certainty of its arguments, and its great number of applications. In metaphysics, on the other hand, no progress can be made ; the disquieting consciousness of its arbitrariness develops and its atmosphere of stagnation and lifelessness is obvious.

The English empiricists, Berkeley and later Hume, held that the idealism of Greek geometry is evident and geometrical constructions are therefore terms without definite meanings. Actually they are nothing but concrete geometrical figures. This view is undoubtedly correct, but it was established in a faulty manner. Concrete geometrical figures are rough and inaccurate, but Greek geometry was based upon the idea of perfection and accuracy. To the present day the adherents of all empirical conceptions of geometry are faced by this difficulty. Kant attacked the problem from a different point of view. His solution was one of genius when the limitations of the science of his day are considered. According to Kant the creative power of Greek geometry lies in the fact that it depends upon *pure intuition*, the peculiar capacity of the mind which permits the relations between geometrical constructions to be perceived independently of experience. He regarded the axioms of geometry as verbal formulations of clear observations which obtrude upon man with inexorable force.

They are called *synthetic apriori judgments.*

Kant's enduring achievement was his discovery that the reasonings employed in Greek geometry are only apparently precise. Every high school student has had ample opportunity to become convinced of this fact. I myself feel that no discipline is more confusing than high school geometry. It is a strange conglomeration of fragmentary reasonings, experience, and conventions accepted without criticism. Its cardinal defects are that it is possible to become thoroughly imbued with it and that in spite of all its defects it functions perfectly. It is clear that it gives strong support to all established prejudices and skilfully concealed paralogisms.

High school geometry requires the acceptance of Euclid's famous postulate:

Through a point outside a line one and only one parallel to this line can be drawn.

High school students have no concept of parallel lines. Parallel lines might be like the lines in a notebook or railroad tracks or perhaps like two infinitely extended perpendiculars. But infinitely extended perpendiculars intersect at the centre of the earth. How then do we know that the lines in a notebook or railroad tracks never meet?

But the high school student does not think very long about this matter but soon passes to the proof of the theorem, if two parallels are cut by a transversal the corresponding angles are equal. If two parallel lines and the perpendiculars to them are drawn it can be seen immediately that the corresponding angles are equal because they are right angles. But this intuition soon proves to be fallacious. The teacher says that merely because the angle at the top is a right angle, it does not follow that the angle at the bottom is also a right angle. This is an unpleasant situation. If such an elementary intuition is fallacious, everything might be fallacious.

Finally an appeal is made to Euclid's postulate. This time the postulate will be considered further. The point involved is, that through a point outside a line two parallels cannot be drawn to this line; but actually this has never been questioned.

It never enters our minds that by means of these clear reasonings we are actually settling the problem of the behaviour of straight lines at infinity, even though we have no idea what this infinity is. But it is the concept of infinity which is at issue, because if the lines met at the centre of a polar star, or if their distance from the point of intersection were $10^{10^{10}}$ light years, they still would not be parallel. Corresponding angles which can be measured were long thought to be equal. However, even if it were impossible to show the slightest difference between them, such lines still would not be parallel. Lines can only be parallel if they never meet. A consideration of these facts is frightening. It is therefore curious that it has never occurred to anyone to frighten the public by pointing them out. Copernicus, Lobaczewski, Lamarck, and later Darwin, were frightened by them, as were Einstein and the proponents of wave mechanics. Nevertheless the geometry of the ancient Greeks is still regarded as the epitome of harmony, serenity of spirit, and what is even worse, of *finitism*.

The difficulties which arise in connection with corresponding angles are soon forgotten. Triangles and polygons are then considered and congruent and similar figures are discussed. But this material is as easy to understand as a primer and does not disturb our serenity. It is not pleasant to think that the difficulties involved in the discussion of parallel lines are of fundamental importance here or that these new results depend upon the concept of infinity.

But it is worth considering that to provide the rural geometer with the constructions required to divide land an adequate concept of infinity is necessary. Everything achieved by the astronomers seems in comparison a trifle and without significance.

In high school the study of the foundations of geometry reduces to the acceptance of Euclid's postulate and the other axioms as obvious truisms. If this view of geometry is maintained, it is impossible to remove metaphysical fear, i.e. the fear of the mystery of existence which was mentioned by Witkiewicz [1] in his remarkable literary works.

It makes no difference whether the thing in itself, or the Kantian pure observation of the forms of phenomena, is employed. All this goes far beyond the bounds of sound reason. Consequently one should not rely upon oneself, but entrust one's fate to a higher power and submit to authority. Another course is open only to stubborn, headstrong maniacs

[1] Stanisław Ignat Witkiewicz: *Nienasycenie. Powieść*, Warszawa, 1930, tom I, p. 78.

like Mikolai Lobaczewski or the Hungarian noblemen, the Bolyais, father and son, who succeeded in destroying the authority of Euclid's postulate. They opened up entirely new perspectives in the study of the foundations of geometry.

2. The fundamental concepts of geometry developed in another very important direction as a result of the work of Poncelet and Brianchon [1] in projective geometry. Projective geometry is concerned only with the mutual position of points, lines and planes. Their metric properties are disregarded.

In 1826 Gergonne [2] formulated the famous principle of *duality*. In conformity with this principle in projective geometry to every theorem there corresponds a certain new theorem, which is obtained from the given theorem by interchanging the role of points and straight lines. In the projective geometry of solids there is a similar correspondence between points and planes.

For example, the theorem: *two straight lines intersect at a point* corresponds to the theorem: *two points determine a straight line.* (The conventional construction: *point at infinity*, is added in order to avoid special cases and it is said that parallel lines intersect at this point.)

This principle was of great methodological significance. It showed that such fundamentally different constructions as straight lines and points, or planes and points have varying roles in geometry. It is therefore clear that the particular properties of these constructions are not of primary concern, but rather the fact that they are merely the auxiliary means used in setting up certain relations which are independent of them.

In 1830 J. Plücker formulated a *principle of infinitely possible interpretations of abstract geometry.*[3]

He pointed out that either a point in space, a straight line, or a circle in a plane may be characterized by the three numbers (x, y, z).

For example $(-1, 3, 2)$ characterizes either the point in space with the co-ordinates $-1, 3, 2$, or the straight line:

$$-x + 3y + 2 = 0$$

in the plane, or the circle:

$$(x + 1)^2 - (x - 3)^2 = 4$$

[1] Cf. Federigo Enriques *The Historic Development of Logic*, translated by Jerome Rosenthal, New York, 1929, p 121.

[2] Cf. Enriques: *l c*, p. 113.

[3] Cf. Enriques: *l.c*, p. 123.

Plücker dealt a mortal blow to geometrical idealism. However, it was not fully appreciated until much later.

In the second half of the nineteenth century Hoüel [1] proposed a formalistic conception of geometry. He regarded geometrical concepts as devoid of intuitive content and viewed points, lines, and planes simply as objects which satisfy axioms, and about which nothing further can be said. He could develop such a view only because he had mathematical formulae at his disposal. Without such formulae no progress could be made, since, as has been seen, the system of geometry at that time was an imperfect apparatus with which it was impossible to work without supplementing it by intuition.

Hilbert took up and solved problems which depend upon a perfectly worked out system of euclidean geometry, but he also showed that strictly speaking all this work was superfluous.[2]

Like Hoüel Hilbert regarded points, lines, and planes as certain undefined objects, of which it is known only that they satisfy certain explicitly stated axioms.

However, Hilbert realized that such definitions are not in themselves sufficient, because the axioms might be contradictory. In that case it would be impossible to speak meaningfully of objects which satisfy these axioms. A *proof of the consistency* of the axioms must be given if this difficulty is to be eliminated. Hilbert's proof consisted simply in interpreting them algebraically. If points are interpreted as three real numbers, planes as linear equations in three unknowns, and straight lines as pairs of such equations, these constructions satisfy the axioms. If the axioms should lead to contradiction there would also be a contradiction in algebra. If the latter possibility is regarded as precluded, there will be no contradiction in geometry.

The work of Hilbert seemed to indicate the triumph of idealism. At first sight it might be thought that this work proved the existence of ideal objects : points, lines, and planes, that the defects of geometry had been removed and that geometry had become a perfect system. It seemed as if idealism were passing through a period of regeneration.

Exactly the reverse took place; Hilbert's method was not adequate to deal with all contemporary geometry. The perfection of his system proved to be illusory when with the

[1] Cf Enriques . *l c.*, pp. 145–6.

[2] D. Hilbert · *The Foundations of Geometry*, English translation by E. J. Townsend, 2nd ed., Chicago, 1921, " Conclusion," pp. 126–132.

development of logic the criteria of scientific accuracy were made more precise. In consequence geometry was transformed into the system of rational metamathematics.

The details of this development will be considered later.

3. From the point of view of methodology non-euclidean geometry was the most important discovery in the history of the exact sciences.

The question to be decided is whether in theory it is possible to construct a square. Squares which are constructed in experience are only approximately accurate. How do we know that in a square in which three of the angles are right angles, the fourth angle may not be less than a right angle?

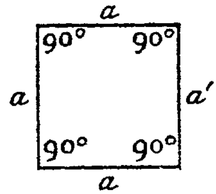

FIG. 7.

This question is usually decided on the basis of Euclid's postulate which affirms that in a plane through a point outside a line, only one line parallel to this line can be drawn. But this postulate was seriously questioned even at the time of the Greeks. In 100 B.C. Posidonius [1] had tried to reduce it to the supposition, that in a plane the locus of points, equidistant from a given straight line is a straight line, but it is clear that this supposition is entirely arbitrary.

In the thirteenth century, Nasîr-Eddîn [2] maintained that Euclid's postulate was evident and he derived it from the premise that the sum of the angles of a triangle is equal to two right angles. The long series of attempts to reduce the contradictory of Euclid's postulate to absurdity began during the seventeenth century. Gerolamo Saccheri,[3] a Jesuit, made such great advances that unintentionally he derived a series of fundamental theorems of non-euclidean geometry.

In the course of the eighteenth century the problems connected with the foundations of geometry became very disturbing and irritating.

D'Alembert regarded the definitions of straight and parallel lines as both dangerous and scandalous. Lagrange interrupted his lecture to the Academy of Sciences on these questions because he noticed that he had made a mistake in his deductions. Legendre, Carnot, Laplace, Fourrier, Monge, and Gauss did

[1] Roberto Bonola: *Non-Euclidean Geometry*, translated by H. S. Carslaw, 2nd ed., Chicago, 1938, p. 3.
[2] Bonola: *l.c.*, p. 10.
[3] Bonola: *l c.*, p. 22–44.

a great deal of work on this subject but did not obtain successful results.[1]

German scholars would like to attribute the discovery of non-euclidean geometry to Gauss. They cite his spoken utterances and his letters to Farkas Bolyai, the father of János, one of the creators of this geometry.[2]

These tendencies were motivated by two factors, patriotism, which is easily understandable, and the prejudice that the greatest discoveries are generally made by acknowledged authorities. The following fact disproves such a hypothesis. Farkas Bolyai devoted his whole life to the problem of parallels and as he himself wrote to Gauss he wasted his life. His sufferings and misery were without limit and would arouse the pity of the most unfeeling and ruthless tyrant.[3] Gauss did not discover even a part of the secret he allegedly possessed. Yet when Farkas' son János discovered non-euclidean geometry, Gauss did not hesitate to assert that he had been occupied with these matters for thirty-five years. This fact is explicable only in the light of Gauss' retiring nature where intellectual discoveries were concerned, which made him discount the value of his views on this subject, and by his lack of the assurance which is necessary in a great discoverer. János Bolyai de Bolya, a young Austrian officer, and Mikolai Lobaczewski, a young professor of the university in distant Kasan, had the requisite assurance.

János Bolyai obtained from his father the idea that the parallel postulate cannot be reduced to absurdity. On the work of his father he wrote as follows:

> "His proof . . . is that everything which contradicts the eleventh axiom can be concealed in infinity. . . ."[4]

This proof, although unsatisfactory, was very suggestive. János Bolyai rightly and profoundly appreciated the fruitful influence of his father. With justifiable severity, he condemned Gauss' claim to priority in the following words:

> "In my opinion and, I am convinced, in the opinion of all unprejudiced men, the reasons adduced by Gauss, as to why he

[1] Cf. Ferdinand Gonseth: *Les Fondements des Mathématiques*, Paris, 1926, p 78

[2] Cf. Ernst Mach: *Erkenntnis und Irrtum*, 5th Aufl, Leipzig, 1926, p 409.

[3] Cf. *Der Briefwechsel zwischen C F. Gauss und W. Bolyai*, herausgegeben durch F. Schmidt und P. Stackel, Leipzig, 1899

[4] Wolfgang und Johann Bolyai *Geometrische Untersuchungen*, herausgegeben von P. Stackel, I Teil, Leipzig u Berlin, 1913, p. 60.

did not wish to publish any of his work on this subject during his lifetime are weak and vain, because in science as in actual life, one's concern should always be to clarify those things which are necessary and generally useful, although not yet clear, and to awaken, strengthen, and expedite the as yet non-existent or rather slumbering sense of truth and right. . .

"And the fact that unfortunately among mathematicians and even among famous mathematicians there are many superficial people, cannot be regarded as a reason why any rational being should produce superficial and mediocre work, and to leave science in a state of inherited lethargy." [1]

János Bolyai was born in 1802. He obtained the foundations of non-euclidean geometry before 1825, and published them ten years later in the supplement to the work of his father which was entitled *Tentamen*. . . .

A few years earlier, Mikolai Lobaczewski published his work and thus robbed János Bolyai of the right to priority.

Even before Lobaczewski, Taurinus had published a work entitled *Geometriae prima elementa*, which, while it contained the bases of non-euclidean trigonometry still did not solve the problem definitively. Lobaczewski and János Bolyai were the real discoverers of the new geometry. Their achievement can be compared only with that of Copernicus. Their greatness lay not only in the acuteness of their reasoning and their power of imagination, traits not peculiar to them, but in their sincerity of purpose and their mastery of their fear of this great new truth.

In 1844 János Bolyai learned of Lobaczewski's work from his father. At first he did not want to believe that the ideas of Lobaczewski could be so similar to his own and he utterly denied this possibility. But the discovery that Gauss was enthusiastic over Lobaczewski's work was the final blow in the tragedy of the Bolyais.

Lobaczewski probably never knew of the work of the Bolyais.

Lobaczewski was born in 1795 in the province of Nizhny-Novogorod. He was the son of Ivan Maximowicz Lobaczewski, a provincial geometer, who had come there from Poland. Some of Lobaczewski's family still lives in Poland.

In contrast to the Bolyais, Lobaczewski was a strong well-balanced man, who was capable of superhuman work. From

[1] W. und J. Bolyai: *l.c.*, p. 96.

1816 on he was a professor at the University of Kasan, to which in various capacities he devoted his entire life.

In 1826 he presented his system of non-euclidean geometry to the Physical-Mathematical Section of the University of Kasan. In 1829 he published the first part of his work *O nachalakh geometrii* [1] in the *Gonets Kazanski*.[2] Thus in the words of Engel [3] was settled a dispute which had lasted two thousand years.

Lobaczewski's work was done independently, since he was cut off from all which had been said on this subject outside of Kasan. It is true that Bartels, his teacher and predecessor in the chair of mathematics at the University of Kasan, corresponded with Gauss, but he was not concerned with the problems centering about Euclid's postulate. He regarded Lobaczewski's ideas as mental exercises and not as scientific achievements. It is well to consider for a moment the effect of the master's opinion upon his pupil.

If the history of science were examined in greater detail, it would be seen that a great number of fruitful ideas have come to naught because the authorities have taken such an attitude. The line of thought taken by ambitious workers, who are eager to work on significant problems and do not wish to waste time on fruitless investigations, has been and still is regarded as useless and wasteful. Such students are encouraged not to work on disquieting problems which stimulate intensive thinking, but to remain in the sterile domain of human thought in which mediocre work flourishes. I know scientists who, on the grounds of *fair play*, quite consciously reject all attempts to solve old problems with the help of a new apparatus of concepts. They prefer that science should stand still and make no progress rather than that a pattern other than the one to which they are accustomed should be employed.

For thinkers of this type the geometry of Lobaczewski and Bolyai has always been and even to-day is unpleasant and unwelcome.

I will not consider here the arguments which have been directed against non-euclidean geometry. They merely show a superficial knowledge of the subject. I only wish to mention the fact that Meinong, the Austrian philosopher whose views were fashionable in Poland before the war of 1914, was firmly

[1] *Principles of Geometry*. Cf. Nikolaj Iwanowitsch Lobatschefskij : *Zwei geometrische Abhandlungen*, Leipzig, 1899, pp. 1–66, translated by F. Engel.
[2] *Kasan Messenger*, cited as *Kasan Bulletin* in Bonola ; *l.c.*, p. 85.
[3] Cf. Lobatschefskij : *l.c.*, p. 381.

convinced that non-euclidean geometry deals with intersecting parallels and in all seriousness attacked this absurdity.[1]

> "The hidden motive for the fact that a euclidean system always required an assumption which was obviously logically incomplete lay in the aversion to speaking of undetermined peripheral distances—an aversion which corresponded to the historical and cultural approach of mankind toward the physically-measured and limited earthly life. Actually all physico-spatial measurements are given from an ethereal distance—from an 'infinite distance'. The moment one ventures into this anew the system becomes lucid and clear." [2]

I think that sooner or later reasoning of this type will involve mankind in hopeless obscurity.

The foundations of non-euclidean geometry will now be developed, because in the light of the means at our disposal such a procedure will not be difficult and it will indeed be worth while, since it will aid our efforts to combat the ever-increasing prejudices of our day.

4. Beltrami,[3] the Italian mathematician, showed that there exists a certain surface—a pseudosphere—upon which the shortest distance between two points behaves like the straight line of Lobaczewski and Bolyai. However, this surface does not give a model of their entire plane.

A model of this plane which is simple and easy to grasp intuitively was given by the late Felix Klein,[4] who was for many years a professor at the University of Göttingen.

If it be imagined that two dimensional beings live within the circle K of radius 1, and if further it be supposed that they contract as they approach the circumference of the circle, which, however, they never reach, from their point of view the points of the circumference of this circle are at infinity. To them a chord of the circle K will be a straight line. Two chords which intersect on its circumference will be

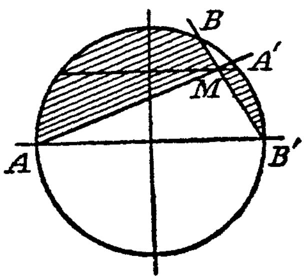

FIG. 8.

[1] Cf. A. Meinong: *Über die Stellung der Gegenstandtheorie im System der Wissenschaften*, Leipzig, 1907, p. 80.

[2] G. A. Kaufmann: *Strahlende Weltgestaltung*, Dornach, Schweiz, 1934, p. 32.

[3] Cf. Hermann Weyl: *Space-Time-Matter*, translated by Henry L. Brose, London, 1922, p. 93.

[4] Weyl: *l.c.*, p. 80.

parallel lines. The pencil of lines which pass through the point M determined by the lines $A A'$ and $B B'$ will have no points in common with the line $A B'$. Under these conditions it may be supposed that beings who live in the circle K define the lines $A A'$ and $B B'$ as *lines parallel* to the line $A B'$. It is clear that the other lines of the pencil would be of such a character that as these beings move along them in the same direction, their distance from the line $A B$ increases without limit.

If now rectangular Cartesian co-ordinates are accepted, and the centre of the model is placed at the origin of these co-ordinates, the radius of the circle-model will be regarded as the unit of distance. Then the equation:

$$x^2 + y^2 = 1$$

will be the equation of this circle-model.

It is clear that the numbers x and y, where

$$|x| < 1 \text{ and } |y| < 1,$$

determine the position of the points within the model.

Now let two beings be imagined at the points designated on the model by the number-pairs $(x_1 y_1)$ and $(x_2 y_2)$ and let the distance d between the points of the model be determined by the well-known formula:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

This distance is not in itself of any great interest. It is a distance designated by us, of which these fictional beings can have no conception. It is, however, of interest to give a formula which would permit us to predict the distance which these fictional beings specify as the distance between the points designated by the number pairs $(x_1 y_1)$ and $(x_2 y_2)$.

Let the expressions $1 - x_1 x_2 - y_1 y_2$ be denoted by the expression $\Omega\,(x_1 y_1 x_2 y_2)$ and the expression

$$\Omega\,(x_1 y_1 x_2 y_2)^2 - \Omega\,(x_1 y_1 x_1 y_1) \;.\; \Omega\,(x_2 y_2 x_2 y_2)$$

by $\Delta\,(x_1 y_1 x_2 y_2)$.

If δ is the distance sought we will have

$$\delta = \tfrac{1}{2} \log \frac{\Omega\,(x_1 y_1 x_2 y_2)^2 + \sqrt{\Delta\,(x_1 y_1 x_2 y_2)}}{\Omega\,(x_1 y_1 x_2 y_2) - \sqrt{\Delta\,(x_1 y_1 x_2 y_2)}}*$$

The method by which this formula was derived will not be given here. It may be thought that it was divined. What is

* Cf. Weyl: *l.c.*, p. 82.

of importance here is that it gives us the solution of the problem.

Its worse defect is that it seems rather mysterious because it is very long and because it contains a logarithm, which is not the common logarithm with the base 10, for which tables have been drawn up for school use, but the so-called natural logarithm having a certain transcendental number e as its base. However, it will be seen that this is a minor matter. This formula, such as it is, permits us to adjust ourselves to the situation without knowing the number e and without working with logarithm tables.

First the distance of the point $(x_1 y_1)$ in the plane (L) from the origin of the co-ordinate system will be computed. In this case $x_2 = y_2 = 0$. If the distance sought is denoted by δ_1,

$$\delta_1 = \tfrac{1}{2} \log \frac{1 + \sqrt{x_1^2 + y_1^2}}{1 - \sqrt{x_1^2 + y_1^2}}.$$

If therefore on the model we draw a circle with radius $\sqrt{x_1^2 + y_1^2}$, it will be represented in the plane (L) by the circle with radius δ_1.

If the radius $\sqrt{x_1^2 + y_1^2}$ approaches 1, δ_1 decreases without limit. The logarithmic number increases without limit, because its numerator approaches 2 and its denominator approaches 0. But if the logarithmic number increases without limit, both the logarithm and one-half of the logarithm also increases without limit.

Thus that which seems to us to be a circle with radius 1 appears to these fictional beings to be an infinite plane.

By a simple calculation it can be ascertained that the formula satisfies the following conditions :

$$1. \quad A A = 0$$

$$2. \quad A B + B C = A C$$

when on the straight line $A C$ the point B lies between the points A and C.

$$3. \quad A B + B C > A C$$

when the point B is not contained in the line $A C$.

The geometrical locus of points whose distance from the x-axis is equal to the given number δ in the plane (L) will now be determined. If it is supposed that the point $A(0, y_0)$ satisfies the desired condition,

$$\delta = \tfrac{1}{2} \log \frac{1 + |y_0|}{1 - |y_0|} \qquad (1)$$

If now an arbitrary point $C(x, 0)$ on the x-axis is considered and if the perpendicular to the x-axis at this point is drawn, and on it a point $B(x, y)$ fixed in such a way that in the plane (L) the distance BC is equal to δ, the chord perpendicular to the x-axis at the point C denoted on the model is the straight line BC which is desired, because since the x-axis is also the axis of symmetry of the model, the angle BCO measured in the plane (L) must equal the angle $B'CO$.

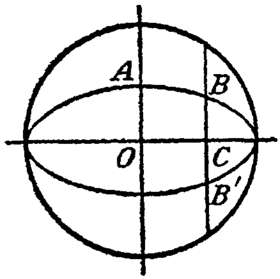

FIG. 9.

If now the formula for distance is applied, the condition:

$$\delta = \tfrac{1}{2} \log \frac{1 - x^2 + |y| \sqrt{1 - x^2}}{1 - x^2 - |y| \sqrt{1 - x^2}} \qquad (2)$$

is obtained. When this equation is combined with equation (1), the result is:

$$y_0 = \frac{|y|}{\sqrt{1 - x^2}}$$

and therefore

$$x^2 + \left(\frac{y}{y_0}\right)^2 = 1$$

Clearly the geometrical locus which is desired appears on the model as an ellipse, whose axes are the co-ordinate axes, where the first semi-axis is equal to 1 and the second to y_0.

This result is of great importance because it removes a fundamental question which presents itself at the very outset. It is seen that if the fictional beings drive telegraph poles, of equal length, along a path represented by the x-axis, in a direction perpendicular to this axis, the tops of these poles will not be on a straight line as in euclidean geometry, but on a curve which is called an equidistant curve.

What would happen if they wish to drive telegraph poles into the x-axis in such a way that their ends would lie on the straight line denoted by the equation:

$$y = y_0$$

will now be considered.

The length of the pole driven in at the point $C(x, 0)$ would be equal to the distance between this point and the point $B(x, y_0)$. If the length of the pole is denoted by

$$\delta = \tfrac{1}{2} \log \frac{\sqrt{1 - x^2} + |y_0|}{\sqrt{1 - x^2} - |y_0|}.$$

If x approaches x_0, where

$$x_0^2 + y_0^2 = 1,$$

δ is seen to increase without limit. An apparently paradoxical situation results. It seems that as the fictional beings move along the x-axis, the telegraph poles become longer and longer. Near the point (x_0, O) the poles would be so long that one would not be able to speak about their reality. At the point x_0 the problem loses its meaning because the poles would have to be infinite in length. Beyond this point the problem is meaningless for the same reason, i.e. because no perpendicular to the x-axis ever intersects the straight line $y = y_0$. It will now be shown that the sum of the angles in a triangle is less than 180°.

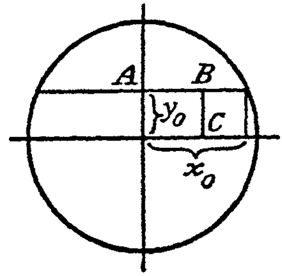

FIG. 10.

Because the x-axis is the axis of symmetry of the circle on the model, a line parallel to the y axis forms a right angle with the x axis in the plane (L). It is therefore easy to understand that in the plane (L) there cannot exist triangles without at least two acute angles. Consequently every triangle can be obtained by compounding two right-angled triangles. It is therefore sufficient to ascertain that in a right triangle the sum of two acute angles is less than a 90°. With this in mind the triangle $O\,B\,C$ is constructed in such a way that the side $B\,C$ is perpendicular to the x-axis and the vertex O is at the origin of the co-ordinate system. If this triangle is rigidly transformed in such a way that the point B becomes the origin of the co-ordinate system and point C lies on the y-axis the vertices of this new triangle will be called O', B', and C' respectively. It should be noted that when $B'\,C' = B\,C$, the

FIG. 11.

point C' must lie above point B or these points could not lie on the equidistant curve. For the same reason point O' must lie to the left of the line $B\,C$. It therefore follows that the line $O\,O'$ lies between the line $O\,B$ and the y-axis and that the sum of the angles B' and O is less than 90°. Because angle B' equals angle B it is seen that the sum of angles B and O must be less than 90°.

The difference between the number 180° and the number measuring the sum of the angles of a triangle in degrees is called the *deficiency* of the triangle. It can be shown that the areas of two triangles are proportional to their deficiencies.[1] I shall observe only that a triangle which is composed of two equal triangles has an area and a deficiency twice as large as the area and deficiency of its component triangles. If the area approaches 0 the sum of the angles of the triangle approaches 180°. This is consistent with the fact that when small areas are involved, the Lobaczewski-Bolyai geometry differs imperceptibly from euclidean geometry.

5. The discovery of non-euclidean geometry was a mortal blow to Kantian idealism. It was proved that there are no such objects as straight lines, either beyond or about us. At best straight lines are certain undefined constructions, which at one time can be regarded as straight lines, at another time as curves. Further investigations by Riemann[2] showed that even straight lines can be regarded as closed curves.

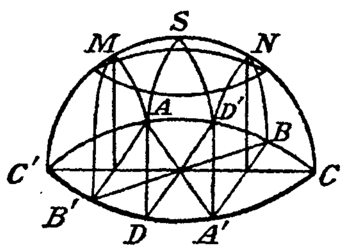

FIG. 12.

If two dimensional beings living on the surface of a sphere are imagined, it can be said that they are the beings seen in glass garden spheres.[3] It should be added that these beings can move over but a small sector of the surface of the sphere. If it is supposed that they measure the distances in accordance with the principles of euclidean geometry, it seems to them that the shortest distance between two points is the arc of a great circle on the sphere. Consequently for them this arc is

[1] Cf. e.g. Henry P. Manning: *Non-Euclidean Geometry*, Boston, 1901, p. 20.
[2] Cf. Weyl: *l.c.*, p. 84 ff.
[3] Cf. Henri Poincaré: *Foundations of Science*, translated by George B. Halsted, New York, 1929, p. 57.

a segment of a straight line and at the same time its length is the length of the segment.

It is clear that an entire great circle on the sphere cannot be regarded by these beings as a straight line since it would then be possible to pass an infinite number of straight lines through two points. These beings will therefore become convinced that the entire plane reduces to a hemisphere and that the points which lie on the same diameter at the base of this hemisphere must be regarded as one point. Consequently in Fig. 12 point A is identical with point A', point B with point B', point C with point C', and point D with point D'. It can be seen that the straight lines have a finite length and are closed. They therefore represent the greatest of all possible circles. If from point $C\,C'$ circles with ever-increasing radii, e.g. the circle $A\,A'\,N\,B\,B'\,M$ are drawn, when the radius increases to one-fourth of the circumference of a great circle on the sphere, the circle becomes the straight line $S\,D\,D'$. It is clear that there can be no parallel lines here and that two straight lines always intersect at some point.

At first sight this may seem like the delirium which accompanies a fever. If, however, we observe that these alarming contingencies can be actualized only at great distances, which these beings cannot possibly reach, these conclusions are much less disturbing. Moreover if the area occupied by these beings were sufficiently small the above constructions would be identical with those of euclidean geometry and this conception might never occur to them. The realization of this fact makes the entire matter clear. The decision of matters which are not confined to the bounds of experience and which one could learn to decide with the help of the conceptual apparatus of Euclid is what is involved. That which was mysterious in Greek geometry has become familiar ; the more successful this adaptation, the more difficult it is to become reconciled to the fact that it might be completely otherwise. In any case it makes no difference which conceptual apparatus we employ provided that it contains no internal contradiction and is compatible with experience. In this way we approach an idealistic position of a peculiar kind, called *conventionalism*.

The theory of John Stuart Mill [1] who regarded points, lines, and planes as hypothetical constructions, chosen in such a way that it is possible to orientate oneself easily in experience.

[1] Cf. John Stuart Mill: *A System of Logic*, London, 1911, Bk. II, ch. 5, p. 147 ff.

Ernst Mach was its actual creator. For Mach[1] science is a conventional apparatus constructed in accordance with the principle of *economy of thought,* i.e. so that it is possible to orientate oneself in experience in the simplest possible way. Just as Kant's theory was the best way to deal with the crisis in the old metaphysics, so was this theory of Mach the most ingenious way out of the new difficulties. It found an ardent propagator in Poincaré,[2] who emphasized the conventional character of geometry with unusual force.

It should be observed that idealism clothed in the feathers of conventionalism became a very dangerous instrument in the hands of those who were reacting against the old dogmatic idealism.

Idealism required a faith which not everyone can produce at will. The doctrine of convenience permits this faith and it was therefore very suitable for the adherents of this doctrine. A large number of them soon appeared. Some began to proclaim *pragmatism,* others *humanism,* others *universalism, panidealism,* and similar doctrines. They desired to construct a picture of the world which would satisfy definite conditions, e.g. such that certain classes which are recognized to be higher would rule the others on the basis of the authority of the exact sciences and maintain their authority at the expense of these other classes. This state of affairs was intolerable. A protest was immediately made against these doctrines. A criticism of their foundations had to be undertaken but required the prior overthrow of scientific conventionalism, whatever the cost.

Riemann took the first decisive step in this direction. He generalized the concept of space and replaced it by the concept of a certain *class of points.* He thus obtained various three-dimensional spaces corresponding to the two-dimensional spaces which have been known for a long time and which can be regarded as surfaces. This conception led Riemann to overthrow the view that the straight line has an exceptional status. Instead of the straight line he proposed the *shortest distance between two points* or the so-called *geodetic line.*

That geodetic lines actually yield the same results as straight lines is easy to understand on the basis of the following example. Weyl writes:

> "If on the other hand we draw figures on a sheet of paper and then roll it up, we shall find the same values for measurements

[1] Cf. Mach: *l.c*, p. 176. [2] Cf. Poincaré: *l.c.*, p. 65.

of these figures in their new condition as before, provided no distortion has occurred through rolling up the paper. The same geometry will hold on it now as on the plane. It is impossible for me to ascertain that it is curved by carrying out geodetic measurements." [1]

If, for example, analogous measurements are made on the surface of a sphere, obviously another geometry will be obtained. It is clear that there are as many metric geometries as there are kinds of surfaces obtained by ordinary bending. Within a sufficiently limited area these geometries differ so little that they are all compatible with experience.

Analogous reasoning can be applied to three-dimensional spaces. However, it is necessary to employ formuli because pure intuition is deceptive here. I regard this factor as decisive. It seems that it is impossible to attain a general concept of geometry without using formuli. It is therefore clear that the conception of geometry as the science of ideal spatial constructions must be nullified.

The preponderance of formuli appears to be even greater, if, following Einstein, it is desired to develop a general conception of *space-time*. It is easy to understand what two-dimensional space-time is. It is sufficient to look at the chart of a recording barometer. Three-dimensional space-time can be explained with the help of a spatial model, but a four-dimensional space-time without formuli would be a confused fantasmagoria. To speak of different four-dimensional space-times it is necessary to employ five-dimensional space-time. It is clear that all this has only as much meaning as do mathematical formuli.

Even though it might be desired to construct an axiomatization of such space-times this would be possible only with the help of an analysis of the formuli previously obtained.

6. At the basis of the problem of time lies the longing for immortality and the fear of death. In general these matters are too close for us to be able to confine ourselves to employing the criteria of sound reason when dealing with them. This element was very strong in the course of the history of human culture and is strong even up to the present day. To the primitive instinct time seems to be an independent state which is like a flowing stream. From most ancient times the tendency to believe that this stream can return to its source has manifested itself. The desire to surmount death is evident in this tendency. On the other hand the conception of an immutable, immovable

[1] Weyl: *l.c.*, p. 89.

eternity in which it would be desirable to locate our soul appears. The positivistically inclined philosopher, Reichenbach, writes :

> "The experience of time seems to be closely connected with the experience of the ego ; 'I am' always means 'I am now' ; but I am in an 'eternal now' means that I experience myself as remaining the same in the flowing stream of time." [1]

This is indeed a pious wish or perhaps rather the privilege of individuals who can easily disregard the discordant changes in what they call their *ego*. It may be supposed that this satisfaction with but little—with regard to observation of facts—together with excessive pretensions to eternal bliss, led to the mythological phantasies of Parmenides, Plato, and St. Augustine. They oppose the notion of the stream of time and seem to find an element independent of time in the fact that they can speak about time. They do not observe that the process of speaking must occur in a time which while it may be a time of higher order, never can be regarded as anything fundamentally different from time.

Parmenides conceived existence as a motionless sphere. The Pythagoreans supposed that this sphere was surrounded by eternity from which the ether is derived. Time formed the boundary line between the finite world and eternity.

Plato and Aristotle made time a property of things. Pawlicki writes :

> "Both agreed that time is inherent in things and above all in their motion and changes, but Plato lays greater stress on motion, Aristotle on its measure, and he regards time as 'the number of motion in respect of *before* and *after*'." [2]

The attempt to surmount time is evident here. This endeavour reaches its peak in St. Augustine, whose struggle with the problem of time soars to pathetic heights of great drama.

In answering the question : what did God do before the creation of the world? St. Augustine remarked that this question is meaningless because it is impossible to speak about actions if there is no time.[3]

[1] Hans Reichenbach. *Philosophie der Raum-Zeit-Lehre,* Berlin, 1928, p. 131.

[2] X. Stefan Pawlicki : *Historja filozofji greckiej* (*The History of Greek Philosophy*), Kraków, 1890, p. 216. Cf. Aristotle : *Physica,* translated by R. P. Hardie and R. K. Gaye, Oxford, 1930, Bk. IV, ch. 11, 219b, line 1.

[3] St. Augustine : *Confessions,* translated by E. B. Pusey, Book XI, ch. (xi, xii) 13, 14, pp. 260–1.

From the point of view of methodology this answer is a great step forward because it is the first critique of the problem in the name of the concept of meaning, which did not involve complete scepticism. Yet the answer of St. Augustine is unsatisfactory because, in order to create time, another time in which the act of creation is performed is necessary. St. Augustine maintained that only present time exists. There is no past or future time. If the present time were to persist it would be eternity. But because it passes away it tends not to be. It is not possible to say that past or future time is long or short, but only that it was or will be long or short.

> "And yet, Lord, we perceive intervals of times, and compare them, and say, some are shorter, and others longer. We measure also how much longer or shorter this time is than that ; and we answer, 'This is double, or treble ; and that, but once, or only just so much as that.' But we measure times as they are passing, by perceiving them ; but past, which now are not, or the future, which are not yet, who can measure ? unless a man shall presume to say, that can be measured, which is not." [1]

The idea of eternal recurrence was the second device employed in surmounting time. The Pythagoreans suggested this idea in connection with what they called the starred year.[2]

This idea is found in a very interesting form in the works of Origen, who combined it with that of a sequence. Origen sought to reconcile the contradiction between the doctrines of the eternal damnation and the highest divine mercy. He sought to remove this contradiction with the help of the conception of successive quasi-recurrences of worlds which in a following world makes possible the redemption of those doomed to eternal damnation in a preceding world. If damnation were eternal it would be possible to suppose that particular worlds last infinitely long ; but it is doubtful whether Origen conceived the matter precisely this way. However, it is certain that Origen conceived subsequent worlds as more perfect than preceding ones, for he explicitly wrote :

> "We think, indeed, that the goodness of God, through his Christ, may recall all His creatures to one end, even His enemies being conquered and subdued." [3]

[1] St. Augustine : *l.c.*, Ch. (XVI) 21, p. 264.

[2] Cf. Zygmunt Zawirski : *Wieczne powroty światów* (*The Eternal Recurrence of Worlds*), Kraków, 1927.

[3] Origen : *De Principiis*, translated by Rev. Frederick Crombie, Book I, ch. 6, PP. 1, p. 54. *Anti-Nicene Christian Library*, vol. 10, Edinburgh, 1869.

He later showed that in this case subjugation can mean only the relationship to God of the Apostles, the Saints, and all those who have followed the path indicated by Christ. It therefore follows that in the course of subsequent times everyone, not excluding devils, will in turn be redeemed.

I think that this conception is no more naïve than the futile efforts expended on this theme in modern times. In fact it is much more profound and consistent than they are

If we examine the history of time more closely and compare it with the history of the concept of space, a marked difference may be noted between them. In the latter use was made of the ideal perfect construction of Euclid, while in the former non-euclidean constructions were employed from the very beginning. In the history of geometry, centuries passed before Lobaczewski made his great contributions ; in the history of time centuries passed before the attempt was made to include time in the euclidean framework.

It is clear that when this was done, time like space became something absolute. Mechanics, like geometry, soon became an apriori science. Henceforth time shared the fate of space. The concept of time like that of space was influenced by the Kantian conception of the apriori form of pure observation. To-day finally time is analysed into separate temporal experiences in addition to which there is only the mathematical apparatus which serves to order them.

Experience teaches us that certain events are earlier and others later, just as there are events which are nearer and those which are farther. Experience enables us to assign certain numbers to temporal events. This, as is known, is done with the help of a clock. Events are not something limited and independent. They are artificially abstracted from the totality of experience, which itself cannot be grasped.[1]

We can call this process formalization. We assign to an event included in a pattern a certain class of numbers, which is called its *spatial representation* and a particular number which is called its *temporal point*.

Obviously it is impossible here to speak of cataloguing events and providing them with labels. Only the construction of an apparatus which, like a counting machine, enables us to predict the results of experience is desired. Just as the fishnet differs from the fish it catches, and the ship from the quay

[1] Cf. A. N. Whitehead : *An Enquiry Concerning the Principles of Natural Knowledge*, Cambridge, 1925, PP. 16.

to which it is fastened, this apparatus differs from the events which it predicts. But just as the net can evoke the image of a fish and the ship the image of a quay, our system evokes certain images of reality. However, it must be kept in mind that these images cannot be grasped and are confused. While they can be employed as auxiliary devices they are not to be trusted absolutely. All confusion concerning the concepts of the physicists results from the fact that these images are taken seriously. Eddington distinguishes the physical from the mathematical concept of a vector. He points out that in mathematics the number triad (X, Y, Z) which characterizes a vector in some co-ordinate system, differs from the number triad (X', Y', Z') which is obtained by a change of co-ordinates. Eddington writes :

> " So far as the mathematical notion of the vector is concerned, the quantities (X, Y, Z) and (X', Y', Z') are not to be regarded as in any way identical ; but in physics we conceive that both quantities express some kind of condition or relation of the world, and this condition is the same whether expressed by (X, Y, Z) or by (X', Y', Z'). The physical vector is this vaguely conceived entity, which is independent of the co-ordinate-system, and is at the back of our measurements of force." [1]

If we permit such a confused representation which includes something besides the results of experiences, obviously we become involved in very dangerous metaphysics. The illusion that metaphysics is a sad necessity to which we must submit, is concerned here. This illusion results from a gross misunderstanding. Physicists do not seem to know that mathematics has long since ceased to be the science of numbers and that the construction of expressions which characterize what is common to all number triads that can be obtained from each other by a transformation of the co-ordinate system, is child's-play for mathematicians. Mathematics supplies us with concrete objects which make all metaphysical representations superfluous.

The following example shows that the concept of *physical quantity* leads to such misunderstandings. Schrödinger regards the velocity $\frac{x_2 - x_1}{t_2 - t_1}$ as a physical quantity, but the derivative

[1] A. S. Eddington : *The Mathematical Theory of Relativity*, 2nd ed., Cambridge, 1924, p. 47.

$\frac{dx}{dt}$ as a mathematical fiction invented by Newton,[1] because he thinks that in physics nothing corresponds to the process of "making" the moments t_2 and t_1 coincide in the limit. Professor Schrödinger does not seem to know that in general it is impossible to make two different moments coincide by successive transition. To do so would require an infinitely long time. The assigning of limits is a certain mathematical operation, as unlike physical processes as is division. The physical quantity $\frac{dx}{dt}$ is unquestionably a fiction since it does not occur in any physical process. However, the quotient $\frac{x_2 - x_1}{t_2 - t_1}$ is also a fiction because just as nature does not perform the operations of the differential calculus it does not perform arithmetical operations. The velocity $\frac{x_2 - x_1}{t_2 - t_1}$ is an expression equally as good or bad as the expression $\frac{dx}{dt}$. What is important is to know how to use both of these expressions. If it is supposed that physical phenomena and time are both discontinuous, the derivatives would still remain meaningful.

The idea that the time of experience is discontinuous is very old. It is encountered as early as the Alexandrian mystic Jamblichus.[2] Ernst Mach wrote:

> "That time and space represent only an apparent continuum and in all probability are composed of discontinuous but not sharply distinguishable elements."[3]

Mach uttered this idea with great timidity undoubtedly because he constantly had in mind the Kantian conception of the apriori form of pure observation.

If it is considered that to-day we are provided with an apparent temporal continuity of events by the movies, it is easy to understand that the conception of Mach may appear entirely natural. In any case it is impossible to take sensual continuity seriously especially because the meaning of this

[1] Cf. Erwin Schrödinger: "Indeterminism in Physics," translated by W. H. Johnston, *Science and the Human Temperament*, New York, 1935, pp. 62–3.

[2] Cf. Pawlicki: *l.c.*, p. 215.

[3] E. Mach: *l.c.*, pp. 447–8.

concept is not known. It has been seen that mathematical continuity has nothing in common with sensual continuity and, as Bergson remarked a long time ago, in a certain sense is its negation. But it must be kept in mind that considerations of this kind are typical in the domain of metaphysics. The concept of real time and real space is but one description more among the many confused descriptions of reality. Actually there are only spatial events which are verified sooner or later. This is all that may be said.

Physicists cannot get accustomed to this state of affairs. It is indeed difficult to understand that physicists who can use mathematical formuli only in very limited domains wish to regard them as objective laws of nature. As a result they would like for this reason to eliminate from these formuli everything which seems to them to be ballast, because there is nothing in experience which corresponds to it. However, they forget that they have derived their criteria of necessity and ballast from these very formuli and that they have done so in a very superficial manner. For example, the representatives of quantum physics would like to introduce the quantum of distance l_0 and the quantum of time t_0 in such a way that $\frac{l_0}{t_0} = c$ where c denotes the velocity of light.[1] They forget that such quanta would be new myths because they would be neither determinate numbers nor mathematical expressions, nor could they be observed in experience.

Science must take into account two fundamental postulates:

Whatever the cost obscure representations should not be taken seriously and the experimental data of the present day should not be regarded as complete.

It should be recognized that physical representations are of value only in so far as they conceptually abbreviate mathematical formuli. The experimental data thus far acquired is of value only in so far as it can be supplemented in the future.

Not until physicists are willing to take these postulates into account will they extricate themselves from metaphysical chaos.

The recently published paper of Born and Infeld[2] is an

[1] Cf. Léon Schames: "*Atomistische Auffassung von Raum und Zeit,*" *Zeitschrift für Physik*, Bd. 81, 1933, p. 271.

[2] M. Born and L. Infeld: "Foundations of the New Field Theory," *Proceedings of the Royal Society of London*, vol. 144, 1934, pp. 425–451.

important advance in the direction of a precise formalization of physics.

7. For purposes of orientation the foundations of the so-called *special theory of Einstein* will be considered as an example upon which the construction of mathematical representations of reality depends. The authority of this theory is now established and its consequences have deeply penetrated the foundations of contemporary physics. This theory resulted from the overcoming of the prejudice concerning the immovable ether, inherited from classical optics.

It is a fact that in the equations of physics the spatial co-ordinate x and the temporal co-ordinate t—I am confining myself to one-dimensional space—are equally and entirely legitimate. If it should be desired to pass from one temporal-spatial co-ordinate system (x, t) to another (x', t') in such a way that the equation of a light wave should not be changed the following transformations must be employed:

$$x' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}(x - v\,t)$$

$$t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(t - \frac{v}{c^2}\,x\right)$$

where c denotes the velocity of light and v the velocity with which the x'-axis moves with respect to the x-axis. The entire difficulty depends upon the fact that the new time t' depends not only on the time t, but also upon the spatial co-ordinate x. Consequently it has nothing in common with traditional time. It was necessary to possess the intuition of a genius and intellectual courage not possessed by the average scholar, and sometimes not even by the greatest thinkers, to maintain that the function t' is the temporal co-ordinate in the new co-ordinate system.

Hadamard, the famous French mathematician, freely confessed that he did not have the courage to eliminate the exceptional role of the order of absolute time. He writes:

> "What do you expect? Like all my colleagues, I admired the work of the physicists which each day is becoming more extensive, and with this admiration was mingled the respect imposed by the consciousness of my incompetency. I did not sufficiently

understand that physics is the domain of those who, on occasion, dare to criticize the work of their predecessors." [1]

These words of another prominent mathematician are the best measure of the greatness of Einstein's achievement.

Nevertheless Einstein's conception is not free from certain idealistic solecisms, which must be removed. In Einstein's theory the old objective space is abandoned because it depends upon the observer. If two observers are imagined to be at a certain distance from each other, which in the course of time does not change, there exists a spatial co-ordinate system common to both observers. We can perceive no event at a point of space specified in this way at the moment it occurs, with the exception of the event at the point at which we are located. We do not perceive an event at a distance x until the time $\frac{c}{x}$ has elapsed. Consequently if events are very remote we would not perceive them for centuries. But at no moment would we perceive even the slightest event which occurs at a given moment in the set of points of this space. Yet this space is related to us in some special way because it is related to our co-ordinate system. If we begin to move, for a motionless observer this space together with our velocity changes. Two observers at the same point at a given moment *have* different co-ordinate systems simply because Einstein's transformation demands this.

While two events occurring at places far removed from each other can be simultaneous from the point of view of one observer, from the point of view of a second observer one event may follow the other a century later. This is very strange. Ultimately it is possible to become accustomed to this, especially because nothing which collides with ordinary experience is involved here, but the invisible space, which changes with the velocity of the observer, is a kind of legend. There is no way out of this difficulty.

Minkowski, the Göttingen mathematician, attempted to eliminate this difficulty by introducing the concept of a *spatio-temporal world*.[2] For Minkowski the world is the class of world-points to each of which different observers assign a certain

[1] Jacques Hadamard : " Comment je n'ai pas trouvé la relativité," *Atti del V Congresso Internazionale di filosofia*, Napoli, 1924, 5–9 Maggio, published 1925, p. 452.

[2] Hermann Minkowski: *Raum und Zeit*, Leipzig, 1909, reprinted in *Gesammelte Abhandlungen*, Bd. 2, Leipzig und Berlin, 1911, pp. 431–444.

four numbers, called spatio-temporal co-ordinates. It is clear that a very extreme idealism is involved here. Even if the fictional character of space-time is overlooked it is still necessary to take into consideration an infinite number of observers, without whom it would be impossible to orientate oneself in this space-time. It must also be supposed that observers in relation to each other while at rest have synchronized clocks. But if it is taken into consideration that among other things the problems of astronomy, interstellar distances, and periods of many light years are involved here, it is difficult not to affirm that these observers who are absolutely necessary in the construction of spatio-temporal co-ordinate systems are indeed like the *deus ex machina* of the old theatre. I think that if in physics the observer is discussed, it is possible to have in mind only oneself or one's colleagues. If infinitely many observers scattered throughout space-time are discussed the boundaries of physics have been crossed and entry made into the confused fields of metaphysical phantasy.

I am trying here to present a certain interpretation of the conception of Einstein, which, while it may lead to far-reaching formal changes in the construction of the system of theoretical physics, is unusually clear and is free from all idealism.

In order to obtain a theory which involves no hypothesis concerning the propagation of light, the following differential equations must be considered:

$$d_1 s^2 = d_1 x^2 + d_1 y^2 + d_1 z^2$$

$$d_1 s'^2 = d_1 x'^2 + d_1 y'^2 + d_1 z'^2$$

These equations will be denoted by S and S' respectively and will be called euclidean systems of three-dimensional co-ordinates. They will be employed as models of space.

A correlation between the points of S and the points of S' will now be defined which corresponds to the primitive idea of uniform motion.

The following symbols of the system S will now be introduced. The corresponding symbols of S' may be obtained by replacing any letter E by the letter E' in a given symbol, and vice versa.

(I) *Points and Axes*

(*a*) A_0 is a *fixed point* of S.

(*b*) Δ is an *axis* having A_0 as origin and intersecting the path of A_0' in at least two points.

(*c*) A and A_1 are any points on the path of A_0'.

(*d*) P and P_1 are any points of S.

(*e*) Q is any point on the plane π of symmetry of Δ.

(II) *Distances and Projections*

(*a*) $r, r_1, \bar{r}, p$, and q are the distances $\overline{A_0 P}$, $\overline{A_0 P_1}$, $\overline{P P_1}$, $\overline{A\ P}$, and $\overline{A\ Q}$.

(*b*) x, x_1, a, and a_1 are the projections on Δ of the vectors $\overrightarrow{A_0 P}$, $\overrightarrow{A_0 P_1}$, $\overrightarrow{A_0 A}$, and $\overrightarrow{A_0 A_1}$.

(III) *Signals and Time-observables*

Any expression $\uparrow (E'\ E)$ is a *luminous signal* emitted from the point E' at the moment of its meeting with E. The moments of arrival of a signal at A_0, determined on a chronometer placed at A_0, are *time-observables* of A_0. We assume the following correlation between signals and variable time-observables of A_0.

Signal	Time-observable of A_0
$\uparrow (A_0'\ A_0)$	τ_{00}
$\uparrow (A'\ A_0)$	τ_0
$\uparrow (A_0'\ A)$	τ_A
$\uparrow (P'\ P)$	τ
$\uparrow (P_1'\ P_1)$	τ_1

No.	Name	Abbreviation	Expression
IV	*Einstein's Time*	t	$\tau - \frac{r}{c}$
		t_1	$\tau_1 - \frac{r_1}{c}$
		t_A	$\tau_A - \frac{\lvert a\rvert}{c}$
		t_0	τ_0

No.	Name	Abbreviation	Expression
V	*Bridgman's Velocity*	V	$\frac{a}{t_0'}$
VI	*Einstein's Velocity*	v	$\frac{a}{t_A}$
VII	*Fitzgerald's Coefficients*	B	$\sqrt{1+\frac{v^2}{c^2}}$
		β	$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$
VIII	*Galileo's Time*	T	$t - \frac{B-1}{V} x$
		T_1	$t_1 - \frac{B-1}{V} x_1$
		T_A	$t_A - \frac{B-1}{V} a$
IX	*Minkowski's Invariant*	$I(E\,F\,G)$	$(E-F)^2 - G^2$

POSTULATES

For any fixed points A_0 and A_0', for any axis Δ, and for any number V other than 0, we have :

$$(1)\ \tau_{00} = 0$$
$$(2)\ t\ .\ t' \geqslant 0$$
$$(3)\ V\ .\ V' < 0$$
$$(4)\ I(t'\,t_1'\,\bar{r}') = I(t\,t_1\,\bar{r})$$

DISCUSSION

The expression V is the velocity which can be determined by a traveller with the help of his clock and the milestones which he passes on his path. It is independent of any theory of the propagation of light. I found this velocity mentioned in Professor Bridgman's book.[1] This velocity can receive any

[1] Cf. Percy W Bridgman : *The Logic of Modern Physics*, New York, 1927, p. 98.

value. Thus the legend according to which there is no velocity greater than that of light is disproved.

The velocity c seems to be the upper limit of the *apparent* velocities used to describe motions observed at a distance.

DEDUCTION OF LORENTZ' TRANSFORMATION

If A_0 and A_0' are substituted for P_1 and P_1' in (4)

$$t'^2 - \frac{r'^2}{c^2} = t^2 - \frac{r^2}{c^2} \tag{5}$$

is obtained.

If A and A_0' are substituted for P and P' in (5)

$$t_0'^2 = t_A^2 - \frac{a^2}{c^2} \tag{6}$$

is derived.

The analogous formula of S' is :

$$t_0^2 = t_A'^2 - \frac{a'^2}{c^2}. \tag{6'}$$

From (6) and (6')

$$t_A = B\, t_0' \tag{7}$$

and

$$t_A' = B'\, t_0 \tag{7'}$$

are obtained by using (V), (VI), (I), (II), and (2).

If these formulae are applied to (VI) and (VII)

$$v = \frac{V}{B},\ \beta = B \tag{8}$$

and

$$v' = \frac{V'}{B'},\ \beta' = B' \tag{8'}$$

is obtained.

If now A_0' and A are substituted for P_1' and P_1 in (4), and (5) and (6) are employed, by an elementary calculation

$$2\, t'\, t_0' = 2\, t\, t_A + \frac{\bar{r}^2 - r^2 - a^2}{c^2}$$

and

$$t'\, t_0' = t\, t_A - \frac{a\,.\,x}{c^2}$$

are obtained.

Then by using (7) and (V)

$$t' = B\,t - \frac{V}{c^2}x \tag{9}$$

is obtained.

For S'

$$t = B'\,t' - \frac{V'}{c^2}\,x' \tag{9'}$$

Using (9) and (5)

$$r'^2 = c^2\,(B\,t - \frac{V}{c^2}x)^2 + r^2 - c^2\,t^2$$

is derived

Then :

$$r'^2 = (B\,x - V\,t)^2 + r^2 - x^2 \tag{10}$$

For S'

$$r^2 = (B\,x' - V'\,t')^2 + r'^2 - x'^2 \tag{10'}$$

If in (10) A' and A_0 are substituted for P' and P

$$a'^2 = V^2\,t_0^2$$

is obtained.

Then by (V) and (3)

$$\begin{aligned} V' &= -V \\ B' &= B \\ v' &= -v. \end{aligned} \tag{11}$$

Adding (10) and (10′) member by member, by using (VIII)

$$\frac{(B+1)\,x\ - V\,t}{(B-1)\,x' - V\,t'} = \frac{T'}{T} \tag{12}$$

is derived.

The following equations are the result of the elimination of t and t' from (9) and (9′) by means of (VIII) :

$$T' - \frac{B-1}{V}\,x' = B\,T\ - \frac{B+1}{V}\,x$$

and

$$T\ + \frac{B+1}{V}\,x\ = B\,T' + \frac{B-1}{V}x'$$

These equations imply :

$$T' = T \tag{13}$$

Equations (12) and (13) imply :

$$(B+1)\,x' + V\,t' = (B+1)\,x - V\,t\,.$$

Then by eliminating t and t' from this equation with the help of (VIII),

$$x' = x - V\,T \qquad (14)$$

is obtained.

This is Galileo's transformation.

Using (VIII) and (8) in (14) and (8) in (9) the Lorentz' transformation

$$x' = \beta\,(x - v\,t) \qquad (15)$$

and

$$t' = \beta\,t - \frac{v}{c^2}\,x$$

is derived.

It should be observed that x' is independent of the way in which Δ' was fixed. Then Δ' must be the path of A_0. In the same way it may be proved that Δ is the path of A_0'. Since the point A_0 is fixed arbitrarily, it is clear that the path of any point P must be a straight line.

From (10) and (15) it follows that

$$r'^2 - x'^2 = r^2 - x^2.$$

It is clear, then, that on the path of P'

$$r^2 - x^2 = \text{constant}.$$

Therefore the path of P' is a line parallel to Δ. It is also clear that Δ' slides along Δ.

If P is any point on the path of P_1', by (14)

$$\frac{x - x_1}{T - T_1} = V$$

Now $x - x_1$ is independent of the way in which A_0 is fixed. Consequently $T - T_1$ is independent of the way in which A_0 is fixed. It is therefore clear that V is the Bridgman velocity of any point P.

THE TRANSFORMATION OF TIME-OBSERVABLES

In order to be able to interpret (14) or (15) a theory of the propagation of light must be assumed. In order to obtain a transformation which would be a simple description of the facts let t and t' be eliminated from (15) by means of (IV).

If the following abbreviations are accepted :

Abbreviation	Expression
B_1	$B + \frac{V}{c}$
B_2	$B - \frac{V}{c}$
V_1	$V \,.\, B_1$
V_2	$V \,.\, B_2$

we have the following transformation :

$x \geqslant 0$		$x' = B_1 (x - V_2 \tau)$
	$x' \geqslant 0$	$\tau' = B_2 \tau$
	$x' \leqslant 0$	$\tau' = B_1 (\tau - \frac{2}{c} x)$
$x \leqslant 0$		$x' = B_2 (x - V_1 \tau)$
	$x' \geqslant 0$	$\tau' = B_2 (\tau - \frac{2}{c} x)$
	$x' \leqslant 0$	$\tau' = B_1 \tau$

A REMARK CONCERNING THE LAWS OF THE PROPAGATION OF LIGHT

On the basis of transformation (14) it is possible to assume that T is the time-observable which corresponds to the meeting of P' with P, taken in P or in P'. In conformity with the postulates of Professor Zaremba,[1] the common-sense idea of time and classical mechanics can be preserved. Relativity mechanics now appears to be just as much a theory of apparent hyper-space as Maxwell's theory of moving electro-magnetical fields. On the other hand the ideas of quantum theory seem to concern real events.

It will now be assumed that the velocity of light has nothing to do with the concept of a thing travelling,[2] but is the number

[1] Cf. S. Zaremba : " La théorie de la relativité et l'experiencé," *Atti del 5° Congresso Internazionale di Filozofia*, Napoli, 1925, pp. 541–4.
[2] Cf. Bridgman : *l.c.*, pp. 101, 152, 157, 164.

which characterizes the path of a luminous source and its position with respect to the observer. It should be noted that these complications have nothing to do with a theory of hyperspace; they concern only the properties of luminous matter. If it could be proved that our theory is consistent with facts, it would be possible to eliminate idealistic metaphysics from the theory of light. At any rate there is no serious reason for neglecting this possibility.

The following law will be assumed:

If it is supposed that a luminous source is placed at A_0' and the observer is placed at P, the distance of any plane from P which intersects the path of A_0' and which is perpendicular to it is not less than the distance from P to π.

Suppose now a signal is emitted from A_0' at the moment T_A.

It is possible to replace this signal with an apparent signal, emitted from A_0' at the moment t_A and which has the velocity c. Then

$$\frac{P}{T - t_A} = c$$

Now by (VIII)

$$T - T_A = T - t_A + \tfrac{1}{2}\,\delta_v \frac{V}{c}\,a$$

where
$$\delta_v = \frac{2}{B + 1}$$

The velocity $\dfrac{P}{T - T_A}$ of the real signal will be denoted by c_V. We get

$$c_V = \frac{c}{1 + \tfrac{1}{2}\,\delta_v \dfrac{V}{c} \cdot \dfrac{a}{P}}$$

The abbreviation

$$V_q = V\frac{a}{q}$$

will be assumed.

It will be supposed that the observer is located on the plane π.

If $V_Q < 0$, V_Q is the velocity of the source A_0' when approaching the observer Q.

If $V_Q > 0$, V_Q is the velocity of the source A_0' when going away from the observer Q.

The corresponding value of c_V is

$$\frac{c}{1 + \frac{1}{2}\delta_v \, . \, \frac{V_Q}{c}}$$

If $\frac{V}{c}$ is a small number, this value is almost the arithmetical average of c and $c - V_Q$.[1]

If P is placed in such a way that a plane through P which is perpendicular to $\varDelta$, does not meet the path of A_0' (finite motion), the velocity c is the limit of c_V as P increases indefinitely where a remains unchanged.

8. The most interesting application of semantical logic to mathematical physics is Herzberg's construction of the notion of a *sequence of von Mises*. The following elementary example will be discussed. K is a sequence of zeros or units which satisfies the following postulates. The ratio of the number of zeros or units to the number of all the members of a given segment of the sequence will be called the *frequency*. It will be supposed that the limit of the frequencies of zeros or of units exists and is the same for all the sequences contained in K.

It is clear that the sequence K could be interpreted as a set of heads and tails which is as long as we please. R. von Mises showed that his sequence can be employed as the fundamental idea of the calculus of probability. In this manner the antinomies of the ancient theories and the idealistic hypotheses of actual science are avoided.[2] Nevertheless in connection with the conception of von Mises certain serious difficulties may be raised. As a matter of fact the sequence of zeros or the sequence of units of a sequence K always has the frequency 1. Consequently it is impossible to speak about a common limit of frequencies of the zeros or the units of *all* the sequences contained in K.

Herzberg has observed that this difficulty disappears at once if a logic based upon the idea of the pure theory of types is assumed.[3] The idea of *all* sequences contained in K is meaningless and it is possible to speak only about *all* sequences of a given type.

[1] Cf. *P P.R.S*

[2] Cf. R. von Mises: *Wahrscheinlichkeit, Statistik und Wahrheit*, II Aufl., Wien, 1936

[3] Cf. J. Herzberg: "Sur la notion de collectif," *Annales de la Société Polonaise de Mathématique*, tom xvii, 1938, pp. 231–244.

The semantical logic will be assumed and the class of *all* classes S of type $N+1$ of finite sequences of zeros and units will be constructed. These finite sequences will be called *intervals*.

It is seen that any class S can be used as a selection rule.

If any sequence C of zeros and units is given any class S determines a sequence contained in C which is the sequence of members of C which immediately follow those segments of C which are elements of S.

For example, the class of all intervals whose ends are zeros determines the sequence of members of C which immediately follow the members of C which are equal to zero.

In semantical logic a sequence of type N of all classes S of type $N+1$ is constructed. Now Wald has constructed the sequence of von Mises for any sequence of classes S.[1] With the help of this construction Herzberg obtains sequences of von Mises which involve no difficulties.

It should be noted that the simple application of a hypothesis which states that the class of classes S is denumerable is not consistent with classical logic. As a matter of fact the class of all selection rules can never be a denumerable class, unless the idea of classes constructible in a given system is employed. Moreover since our class of classes S is any denumerable class of S, the fundamental operations [2] of the calculus of probability can never be applied to a corresponding sequence of von Mises outside of the set of selection rules.

Wald's construction enables us to have *decidable* sequences of von Mises.[3] However, the sequences of von Mises which occur in practice seem to be undecidable.[4] Thus a profound relation between events which never can be foreseen and undecidable propositions has been established. The theory of Herzberg enables him to assume the following definition of accidental events which is completely different from the irrationalistic conceptions of accident. Herzberg states that accidental events are governed by physical laws whose consequences are undecidable. It should be noted that there is no serious reason to assume that they are undetermined. Thus

[1] A. Wald: "Die Widerspruchsfreiheit der Kollektivsbegriffes der Wahrscheinlichkeitsrechnung," *Ergebniss eines mathematischen Kolloquium* 8, Wien, 1937, pp. 38–72.

[2] Cf. R. von Mises: *Wahrscheinlichkeitsrechnung, Statistik und Wahrheit*, Leipzig und Wien, 1931, PP 3, "Die Grundoperationen," p 73 f.

[3] Cf. Wald; *l.c.*, p. 49 f.

[4] Cf. Herzberg: *l.c.*, p. 243 f.

all modern arguments which seem to overthrow the doctrine of determinism appear to be illusory.

To obtain an example of events which are fully determined yet cannot be predicted, it suffices to consider the algebraic expression of Fermat:

$$p^n + q^n = r^n.$$

This expression can be employed as a pattern of experiments. If arbitrary integers are substituted for p, q, r, and n it can be seen whether or not they satisfy this equation. As no demonstration of Fermat's theorem has ever been given, it is impossible to predict the result of such an experiment for an infinity of cases. However, there can be no doubt that this result is fully determined.

The traditional conception of determinism as a doctrine concerning the predictability of events does not give the essence of the matter. It may be impossible to know what the result of an action will be, but it is possible to know definitely that the result is completely determined.

It should be observed that no general definition of *determinism* can be given, unless metaphysical hypotheses are accepted. But this does not really matter. A general definition of a concept is unnecessary if there are concrete examples which explain it. Since there are examples of determined events, it may be expected that other events are also determined, although a determining law is not given. In other words the concept of *determinism* is simply regarded as a primitive idea, like the concept of truth or the concept of implication.

9. Classical physics was based upon an idealistic conception of reality. For Descartes reality was the world with which Greek geometry dealt. He was of the opinion that reality is known through innate ideas. Locke regarded the laws of nature as generalizations made from experience. Kant returned to apriorism but conceived reality as the world of phenomena and not as a thing in itself. These views do not differ fundamentally. In all of them reality was regarded as something perfect, something determined by exact and immutable law. This conception of reality greatly influenced physicists and was maintained by them until recently. In physics textbooks and in the texts of applied mathematics it was accepted as certain that a real segment has an exactly determined length x which can be measured with the help of a number of approximations. The measured distances x_1, x_2, . . . were opposed to the

real distance x and the error of measurement was defined as the number :

$$x - x_i \quad \text{where } i = 1, 2, \ldots$$

On this view the calculus of probability refers to an ideal reality. It is to be contrasted with *probability* which refers to experimental data. The applications of this calculus to physics were based upon the above suppositions. Classical physics was *deterministic*. Determinism obviously was based upon an appeal to an ideal reality. The reality accessible to experience was determined only approximately but it was supposed that as experimental means develop, it will be possible systematically to make these approximations more accurate.

Indeterminism, the doctrine according to which systematic increase in the precision of approximations is impossible, is the direct opposite of determinism. Both doctrines were based upon a metaphysical view of reality and consequently were equally inconsistent with sound reason. From the point of view of a science based upon sound reason, the problem of determinism and indeterminism must be formulated in an entirely different manner. It must be considered in the light of the following three problems : (*a*) the problem of meaning, (*b*) the problem of prediction, (*c*) the problem of formalization. These problems will now be considered.

(*a*) THE PROBLEM OF MEANING

If it is desired to remain within the limits of sound reason, the confused idealistic doctrines cannot be supported. As Hume and Mach pointed out, it must be said that the ideal length of a segment does not exist. Consequently it is meaningless to discuss it. It is possible to speak only of the numbers which are obtained by measurement. But measurement is a crudely defined activity. From the point of view of measurement slight differences as to results are disregarded, just as from the point of view of semantics the differences between various copies of the letter *a* are disregarded. But the formalization of certain images is involved in the former case. There are no two identical images. Identity is the product of criteria which operate automatically. It is therefore clear that measurement gives no basis for the establishment of a one to one correspondence between the results of measurement and a real number. It is possible to speak only of a one-many correspondence between them. In other words many real numbers correspond to one measurement and the class of real numbers is not precisely determined. The only way to make this statement more precise is to fix the limits between which the number obtained by measurement can vary, i.e. to designate two numbers, between which the number obtained by measurement may be found.

If it is supposed that a and b are such numbers, the physical law of measurement can be represented by the double inequality:

$$a < x < b,$$

where x is a variable representing any number which can be obtained by measurement.

It is clear that this law of measurement permits the prediction of the results of measurement as accurately as the theorems of arithmetic permit the prediction of the results of the operations involved in calculation.

This means that the inequality will always be satisfied by a number obtained by measurement. Approximations and probability are not mentioned here because there is nothing to be approximated to and there is no thing to which the results would stand in the relation of probability. The statement that the results of measurements must satisfy this double inequality is the only truth.

If this state of affairs is fully understood it will be possible to eliminate completely the indeterministic ideas of contemporary physics.

It may be seen that for contemporary physicists the concept of indeterminism is only a way to describe the fact that the old conception of determinism has no meaning which is compatible with sound reason. But from this fact it by no means follows that a determinism which involves no idealistic suppositions would be impossible in physics.

The following theoretical result which was obtained by Professor Heisenberg greatly impressed the physicists. The difference between the limits within which the result of the measurement of the position of an electron can vary will be denoted by Δx. The difference between the limits within which the result of the measurement of the momentum of an electron can vary, i.e. the velocity multiplied by the mass, will be denoted by Δp. By certain theoretical considerations, Heisenberg showed that

$$\Delta x \cdot \Delta p \geqslant h,$$

where h is a constant number, i.e. Planck's constant.[1] Szczeniowski and Ziemecki explain this result in the following words :

> " Let us suppose that a certain measurement permitted us to determine the momentum of an electron precisely. To complete the data required by classical mechanics, we must still measure the position of the electron. We see that the measurement of the position gives us, although with a certain degree of indeterminacy, the position of the electron. But it thereby partially nullifies our previous knowledge concerning the momentum of the electron because of the variation of value which is both unavoidable and indeterminate." [2]

Later it is learned that it would be necessary, for example, to let the electron pass through a narrow slit in order to designate the position of an electron. But in passing through the slit the path of the electron can change direction within certain limits; consequently its momentum becomes indeterminate within certain limits.

Heisenberg's theory is of interest because he fixed certain limits of accuracy for measurements, which cannot be transgressed and because, contrary to the views held up to that time, he discovered a relation between the measurement of position and the measurement of momentum. But it does not follow that his theory yielded something essentially new from the point of view of the problem of determinism. Once it has been confirmed that the concept of ideal length is meaningless and that experience furnishes not determinate numbers but classes of numbers which lie between certain limits, such concepts as *determinism* and *indeterminism* must be adapted to this state of affairs. If constructions based upon idealistic physics are employed, there is obviously some basis for speaking about indeterminism since it will be believed that there exist some ideal position and some ideal momentum of the electron which will never be known. But the point is that this view is erroneous. Indeterminism would then mean that there is a certain arbitrariness in the parade of phenomena. Professor Schrödinger who compares electrons with human individuals and on the

[1] Cf. Szczeniowski i Ziemecki. *Promieniowanie i Materja* (*Radiation and Matter*), Warszawa, 1932, p. 157.
[2] Szczeniowski i Ziemecki : *l.c.*, p. 162.

basis of an indeterministic doctrine tries to secure "the least possible interference in the private affairs of the individual",[1] seems to conceive indeterminism in this way. It is worth adding that this argument is double-edged since it can also be maintained that we proceed more circumspectly at the very times when the laws which govern the life of the individual are taken into consideration. If it were believed that there are no such laws, we might be tempted to create them because of our personal desires.

But if it is affirmed that independent of ourselves we have no concepts of momentum and position, scientific meaning must be denied to the question: what are the real momentum and position of an electron? Consequently it is not possible to talk significantly about the free movement of an electron, where this motion cannot be known.

The situation here is similar to that which arises in connection with matters of daily life. We have no precisely determined concept of *honesty*. Our adjustment in experience may be better or worse in accordance with the way in which this concept is defined. If it is desired that a strict definition of honesty be advanced, hardly anyone will be honest and this conception will be useless. The situation which arises here is just like that which results from the over-extensive desire to make precise the position of an electron. The domain of the concept honesty must then be fixed in such a way that a sufficiently large group of people will be included under it. Similarly in physics a concept of position must be accepted which is broad enough to determine the concept of momentum within sufficiently narrow limits. But just as in the first case, it cannot be said that our abandonment of a strict view does not permit us to show who is *truly* honest, in the second case we cannot lament that we do not know the *true* position and momentum of an electron, simply because these concepts are meaningless.

We can only affirm that events can be predicted within the limits suggested by our apparatus of concepts. The desire to predict events which are not within these limits is metaphysical and utterly inconsistent with sound reason.

Professor Heisenberg pointed out that what is involved here is an alteration of the conditions of experience which is provoked

[1] Cf. Erwin Schrödinger: "Physical Science and the Temper of the Age," translated by Dr. James Murphy, *Science and the Human Temperament*, New York, 1935, p. 132.

by the intervention of the observer.[1] This observation is correct, but may easily lead to just such misunderstandings as arise in connection with the argument for the impossibility of introspection. It is said that introspection is impossible because the course of phenomena changes. Actually introspection enables us to know those phenomena which are accessible to introspection. Phenomena which are not accessible to introspection are simply myths which are a result of the theory of the unity of consciousness. Similarly, beyond the limits fixed by Heisenberg, the momentum and position of an electron are also myths; consequently they cannot be said to be unknowable. The concept of unknowability is closely connected with the supposition of the existence of things which cannot possibly be experienced. If this supposition is not accepted the concept of unknowability must be abandoned.

If it were held that some day it may be possible to go beyond the limits set by Heisenberg, a new conception of momentum and position would then be obtained, but at the same time we would have laws determining the results of the measurements in question. Consequently in no case is there any basis for speaking about indeterminism.

(*b*) THE PROBLEM OF PREDICTION

Outside the domain of obscure formulations where the issue between indeterminism and determinism reduces to the problem of reality, there are concrete problems which put the whole matter in a very different light. The argument of Pascual Jordan [2] should be considered.

Jordan shows that a polarized ray allowed to pass through a Nicol prism, whose plane of polarization forms the angle ϕ with the plane of polarization of the prism, only partially passes through this prism. A fraction $\cos^2 \phi$ of the intensity of light passes through the prism, the fraction $\sin^2 \phi$ is reflected.

If a single light quantum is considered it is not possible to predict whether it passes through the prism or whether it will be reflected. It is clear that, no matter what a light quantum may be, a concrete experience is necessary to determine how it will behave. The result of future experience is simply unknown. Theory gives the probability $\cos^2 \phi$ that the quantum

[1] Cf. Szczeniowski and Ziemecki: *l.c.*, p. 160.

[2] Cf. Pascual Jordan: "Quantenphysikalische Bemerkungen zur Biologie und Psychologie," *Erkenntnis*, Bd. 4, 1934, p. 222.

will pass through the prism and the probability $\sin^2 \phi$ that it will not.

Jordan asserts that there is no way out of this difficulty because infinitely many properties of photons with respect to an infinite number of angles ϕ would be required. This is a misunderstanding. This phenomenon is typical in contemporary physics. On the one hand there is discontinuity and extreme finitism, on the other slavish dependence upon mathematical idealism. But the probabilities $\cos^2 \phi$ and $\sin^2 \phi$ have no absolute meaning, and below certain limits lose all experimental meaning. Furthermore it is not possible to count 10^{10} possible positions of the Nicol prism.

Obviously this remark does not explain the phenomenon and does not pretend to do so; it merely serves to warn against the over-simplification of problems. Investigations of this type are still at an initial stage and it is neither necessary to answer all questions, nor can it be pretended that they have been answered.

I think that it must be explicitly pointed out at once that the fact that it is impossible to predict definite phenomena, does not prove that these phenomena are not determined.

In a paper read at the Congress of German Mathematicians at Königsberg, Professor Heisenberg formulated the doctrine of prediction as follows:

> "If the present state of an isolated system is known in all its determining constituents, it is possible to calculate the future state of the system therefrom." [1]

Heisenberg rejects this formulation because he feels that knowledge of all the determining constituents is impossible. But as has been seen the system of rational metamathematics permits all the construction rules for the theorems of arithmetic to be known and thus places all the determining constituents at our disposal. In spite of this fact it is impossible, as has been seen, to predict all possible results which can be obtained by the application of our rules. Yet the system of metamathematics must be regarded as the model of a determined system. Consequently it must be concluded that prediction is not a decisive element here. Rather it is this fact of the existence of a precise system which is crucial.

This example seems particularly convincing to me because

[1] Werner Heisenberg: "Kausalgesetz und Quantenmechanik," *Erkenntnis*, Bd. 2, 1931, p. 174.

it would obviously be absurd to speak about the indeterminism of addition and multiplication. But it is not difficult to give examples, derived from the natural sciences, which clearly show that it is often impossible to dream of predicting events even though not for a moment is there any doubt that they are determined. The creator of quantum theory, Professor Planck, cites meteorological phenomena to illustrate this point.[1] In this connection I offer the example of a piece of paper floating in the breeze.

(c) THE PROBLEM OF FORMALIZATION

The whole problem, I am convinced, reduces to the question how far reality can be formalized. It is certain that the formalization of reality will never be completed.

If it were supposed, as was done by Meyerson,[2] that there is a certain completed reality with which we become acquainted gradually it would have to be agreed that our formalization is at base some irrational remainder or rather a residue or odds and ends. It would have nothing in common with the *thing for us* [3] about which Engels wrote, and could not influence our life. But this remainder or residue is a typical idealistic fiction and cannot be reconciled with the criteria of sound reason. However, it must be kept in mind that in speaking about reality we have in mind not some ideal object but the patterns which must be employed in dealing with a given case. The problem of determinism and indeterminism can have a meaning which involves no metaphysical solecisms only if these concepts are connected with the concept of formalizing reality. The fact that these patterns never form a completed class causes a lack of belief in the possibility of further formalization and the belief in such a possibility will always be widespread. There is no theoretical argument which can convince an opponent of the unlimited possibilities of development of human thought ; consequently it cannot be proved that determinism is a true doctrine. However, it is possible

[1] Max Planck : "Pojęcie przyczyny w fizyce," *Zagadnienia współczesnej nauki* ("The Concept of Cause in Physics," *Problems of Contemporary Science*), translated by Edward Poznański, Warszawa, 1933, p. 30.

[2] Cf. Emile Meyerson . *Identität und Wirklichkeit*, deutsch von K. Grelling, Leipzig, 1930, p. 310.

[3] Quoted by V. I. Lenin : *Materialism and Empirio-criticism*, translated by David Kvitko, New York, 1927 ; *Collected Works of V. I. Lenin*, vol. 13, pp 75–77.

to combat indeterministic irrationalism successfully. Contemporary physicists, whose formalizations of reality have been carried unusually far, wage just such a struggle. From our point of view contemporary physics is deterministic *par excellence*. If physicists proclaim indeterminism, they do so only because they have in mind not the traditional conception of indeterminism but the struggle with the naïve determinism of classical physics. They do not realize that in this way they are giving a weapon to irrational reaction and indirectly are helping to strengthen the prevailing chaos.

10. I was involved in the following situation: A pupil, whom I asked to illustrate incomplete induction, unhesitatingly replied that Argentine, Brazil, Uruguay are republics and therefore every country in South America is a republic. I was astounded by this completely erroneous answer and asked where the candidate had obtained this example. I learnt that he had taken it from Professor Kotarbiński's *Elements*. Interested by his reply I thoroughly examined the *Elements* and discovered the source of his misunderstanding. In this book the following passage may be found:

> "And here is an example of incomplete induction: Argentine is a republic, Brazil is a republic, Uraguay is a republic; therefore every independent South American country is a republic." [1]

It seems that Kotarbiński does not infer from separate examples, but deduces the conclusion from a general premise. But this procedure has nothing in common with induction. If a general law is known it is obvious that it can be applied to particular cases, but the question at issue is what must be done if such a law is not known. Obviously it must be constructed in some way. If, however, it were constructed in the way advised by Kotarbiński, a methodological error would be committed. If, for example, such reasoning had been carried through during the reign of Don Pedro, the emperor of Brazil, a false view of the matter would have been obtained, which could not be avoided by Kotarbiński's purely formal subterfuge.

Aristotle thought that it follows from the fact that man is long-lived and a horse is long-lived and a mule is long-lived that all bileless creatures are long-lived. This view worried

[1] Cf. Tadeusz Kotarbiński · *Elementy teorii poznania, logiki formalnej i metodologii nauk* (*Elements of the theory of knowledge, formal logic, and methodology of the sciences*), Lwów, 1929, p. 278.

Kotarbiński[1] somewhat, as is shown by the fact that after the word *bileless* he placed an exclamation point in parenthesis. But his anxiety was not so great that it suggested to him the conclusion that the whole doctrine is a common misunderstanding. Actually this method is in general erroneous. In the Old Testament, for example, it led to the false conclusion that hares are cud-chewing animals. It also involved the scientists of the Middle Ages in an unparalleled confusion of concepts. It is difficult to discover why in the twentieth century youths at the Polish universities must struggle over problems which were obsolete even at the time of Galileo and Newton.

Dr. Metallman praised Kotarbiński's stand. He explicitly writes:

> "The thesis that explanation and the inductive method in general depend upon inferring from the particular to the general, is not on the whole correct."[2]

But in another place Dr. Metallman formulates the principle of induction as follows:

> "in nature there are no unique things."[3]

and then supplements this statement by the principle of *partial identity*, in accordance with which

> "if a certain element of nature is repeated still another element is always repeated."[3]

The reader is really in difficulty, because it is difficult to discover what the purpose of these principles could possibly be if not to pass from the particular to the general.

I think that Dr. Metallman's formulations correspond to actual tendencies of thought about nature, but they are of little value because the concepts *unique thing* and *repetition of the elements in nature* are so vague.

Consider the following examples:

If a botanist who is studying the flora of a given region meets an example of a certain species for a second time, this fact will have no significance for him. If, however, he is preparing statistics concerning the plants of this region, this

[1] Cf. Kotarbiński: *l.c.*, p. 278.
Aristotle: *Analytica Priora*, translated by A. J. Jenkinson, Oxford, 1928, Bk. II, 23, 68*b*.

[2] Cf. J. Metallman: *Determinizm nauk przyrodniczych* (*Determinism in the Natural Sciences*), Kraków, 1934, p. 332.

[3] Metallman: *l.c.*, p. 393.

fact will be very important to him. This illustration is based upon the argument advanced by Nicod in his polemic with Keynes.[1]

It should be added here that the investigator in question must be careful not to observe the same example twice. This would not be the case if experimental investigation concerning the psychology of observations of nature were being undertaken. Then each observation must be regarded as an individual fact.

Earlier philosophers of nature were of the opinion that individual facts should not be considered in the study of nature. Poincaré [2] wrote that the landing of John Lackland at a given place is not a fact of nature, because this fact will never be repeated. But neither will the diremption of the earth from the sun ever be repeated, nor will Orkisz' comet ever reappear. It is indeed possible to hold a different opinion on the basis of the belief in the recurrence of worlds which has previously been mentioned, but then the second landing of John Lackland at a given place should also be taken into account.

These considerations lead to the conclusion that no general formula can describe the inductive method. In some cases it is correct to derive the general conclusion from one occurrence. If, for example, an individual has constructed a radio to-day, then it can be inferred that he can build a radio, and that he knows that if he desires he can construct one to-morrow. Similarly if an unexpected effect were obtained in a physics laboratory, it would be expected that it could be repeated in some other laboratory so that local conditions which are difficult to observe, or possible auto-suggestion on the part of the discoverer, might be eliminated. If this effect is obtained in the other laboratory, the matter is definitively settled and no one would doubt that it is possible to obtain the given effect whenever it is desired.

In other cases the individual cannot be so sure that he can derive a general conclusion from one occurrence. If he has in mind a particular experiment which he is about to perform, its success can only be probable because unforeseen obstacles which would thwart his plans might occur.

However, there are cases when even a great number of facts, even all the facts known from the beginning of time, not only do not give an individual the right to infer with certainty or

[1] Jean Nicod: *Le problème logique de l'induction*, Paris, 1924, p. 76
[2] Poincaré. *l.c.*, p. 128.

even with the slightest probability that something will be so. John Stuart Mill [1] pointed out that it could not be inferred that there are no black swans from the fact that all known swans were white.

The history of any discovery illustrates this point. Before the construction of the first aeroplane when all attempts to fly from Daedalus and Icarus down to Leonardo da Vinci had ended in disaster there was no reason to suppose that men would ever fly, even though students of the Scriptures pointed out that the prophet Daniel had explicitly predicted that men would fly. Nevertheless if on the basis of the state of affairs at that time someone had inferred inductively that men would never fly, he would have committed a gross methodological error.

It can be seen that the problem of induction is very difficult and involves in some way the meaning of the concepts concerned, knowledge of the facts and of changes in the relations between them.

The elements of guessing and of making extrapolations which are dictated by emotion cannot be eliminated from inductive reasoning.

Ernst Mach wrote [2]:

> "If our interest in a new fact is aroused by its direct or indirect biological importance and by its agreement or disagreement with other facts, by means of the psychical mechanism of association we concentrate our attention upon two or more related elements in the facts. *Abstraction*, the failure to observe apparently unimportant elements, with the result that the individual case takes on the character of the general case which represents many similar cases, involuntarily occurs. Although the occurrence of this psychological situation is naturally favoured by the accumulation of *several* similar facts, it can be brought about by vital interest in *one* such fact."

The simplest automatically operating process of formalizing reality is involved here, i.e. the very process which in semantics results in the identification of two different copies of the letter *a*.

It is clear that the whole matter reduces to the direct operation of the criteria of sound reason. These criteria cannot be formulated in a pattern and the exact bounds of their operation cannot be fixed. On the one hand habit and routine,

[1] Mill: *l.c.*, Bk. 3, Ch. 3, PP. 2, p. 204. [2] Mach: *l.c.*, p. 315.

on the other hand the construction of more and more perfect apparatus of observation, such as the magnifying glass, the microscope, the telescope, the spectroscope, the Röntgen ray, and so forth, function here. It is known that by means of such apparati what was formerly undetermined and explained by confused phantasies has become entirely clear and even trivial. It is sufficient to consider the history of medicine and the role of the microscope and Röntgen rays in its development. It should also be kept in mind that beside extending the formalization of reality, new apparatus also makes possible the extension of the domain of experiences which have already been formulated in terms of a pattern. Without new apparatus this extension would not be successful, simply because it would never occur to anyone to perform experiments which on the basis of concepts prevailing at the time would have to be acknowledged as pointless. I have in mind here the discovery of the isotopes of lead. In Szczeniowski and Ziemecki's excellent book, the following passage may be found:

> " On learning of the amazing results of Aston, it is hard to rid oneself of a certain scepticism and disbelief. At any rate the method of canal rays is indirect. We might want to have isotopic variations in weighable quantities; we might want to be able to discover differences in atomic weights by ordinary chemical methods, and to convince ourselves by means of a picnometer that the densities of the various varieties of elements differ. This has indeed proved to be possible. Striking results have been obtained in the case of lead. It has been confirmed that in different parts of the terrestrial globe, there are ores which contain two varieties of lead with atomic weights which differ by almost one per cent. Second year chemistry students could discover this difference, which is comparatively great."[1]

If we read the history of any invention, even if very complicated apparatus is employed, we become convinced that the whole matter is simple and can be precisely formulated. In the last analysis every apparatus is something like the system of rational metamathematics. But if we desire to formulate the concepts of the apparatus in a pattern we are faced by an impossible task. All attempts to achieve such a formalization have proved to be fruitless from the time of their discoverer, Francis Bacon, up to the present day. It is the history of inventions alone which is instructive. The

[1] Szczeniowski i Ziemecki: *l c.*, p. 26.

systematic classification of inventions might prove to be of some advantage for further investigations. But what is common to all apparatus is a definite operation. What is really important are the individual characteristics of different types of apparatus. The situation is like that in psychology, except that there is a much greater variety of types in the world of apparatus.

The problem of induction becomes much more complicated if we go beyond cases of a general character and seek to create a series of individual cases. Then the phenomenon, which appears in complete induction, is concerned. Complete induction is required in the natural sciences if, for example, it is desired that the ancestry of a given living individual be traced.

I have mentioned that if the principle is accepted that the mother of a person is a person, it must be concluded that a certain woman (Eve) had no mother, or that mankind has lasted forever. This is a valid argument which depends upon complete induction. While the role of complete induction is negative here, it is, as is seen, very far-reaching. The pattern of complete induction, the ancestral relation, which can be applied to nature, was created by Whitehead and Russell.

In making positive applications of complete induction in the natural sciences difficulties are encountered, because in general the transition from any one case to the following one cannot be confirmed with complete accuracy. It is here that the calculus of probability must be employed in the natural sciences. Only at this stage do the natural sciences lack complete certainty and approximation must be employed. It is obvious that here human and not divine certainty is meant. I know that it is not absolutely certain that stones will fall upon the earth to-morrow, and that they will not change into Raphaelean angels. I know this, but I have explicitly stipulated that I do not intend to take this type of certainty into account.

The law of gravitation and the fact that the letter *a* does not change into the letter *b* are equally certain. Without constant letters there would be no mathematics, and so in general there would be nothing certain. That which is as certain as the fact that letters differ, cannot be regarded as probable ; this postulate cannot be questioned. Only if it is kept in mind can the confusion of concepts in connection with the foundations of the calculus of probability and statistical methods of investigation be avoided.

Chapter X

THE PROBLEM OF REALITY

1. The creation of a precise language for philosophy, using the language of the exact sciences as its model, is a problem which has troubled logicians since most ancient times. Undoubtedly Aristotle's syllogistic logic, Raymond Lull's *Ars magna*, the *characteristica generalis* of Leibniz and Spinoza's *Ethica more geometrico demonstrata* were all attempts to solve this problem. Needless to say these attempts all proved illusory although their authors attributed very great significance to them. How much Leibniz expected of his discovery the following passages give evidence :

> " During my eighteenth year, while writing the little book *De arte combinatoria*, published two years later, I hit upon a certain line of thought, the wonderful secret of true analysis, whose result is language or rational characteristic. I believe that no one else has perceived this, for anyone who had done so would have put all else aside and pursued it since nothing greater can be achieved by any man." [1]
>
> " This is the principal aim of that great science which I have been accustomed to call characteristic, of which what we call Algebra or Analysis is only a very small branch. It is characteristic which gives the words to languages, the letters to words, the figures to arithmetic, the notes to music. . . . Finally it is characteristic which permits us to reason with but little effort by substituting symbols for things in order not to hamper thought." [2]

It is known to-day that the suggestion of Leibniz is only of historic interest. The system of rational metamathematics permits the construction of symbolic representatives of the objects of experience. These symbolic representatives are of the type treated in theoretical physics. This is indeed natural, for the very objects upon which the system depends, namely letters, are just such symbolic objects. However, it should not be forgotten that a whole class of the properties of letters are neglected. In particular there is no way of distinguishing

[1] G. W. Leibniz " Letter to Tschirnhaus," 1679, *Briefwechsel von G. W. Leibniz mit mathematikern*, ed by C. J Gerhardt, Bd. 1, 1899, pp. 405–6.

[2] G. W. Leibniz : *De la méthode de l'universalité*. Cf *Opuscules et fragments inédits de Leibniz*, ed. by Louis Couturat, Paris, 1903, pp. 98–9.

separate copies of a letter. Clearly then it cannot be expected that rational metamathematics will prove fertile where such objects of experience but not their patterns are concerned. Where investigation of individual objects is begun, where a particular copy of the letter *a*, a particular automobile, a particular human organism, a particular society is concerned, the system of metamathematics automatically fails to work. However, there is by no means any foundation for a conclusion such as that of Meyerson that reality is irrational and that it can be rationalized only in part.[1] There is no reason to think that reality is irrational. Rather it is simply to be confirmed that reality is never given as something completed and that only the patterns of reality are treated. For every pattern there is a correspondent. The farther the process of formalization is carried on, the more complicated will the system be, but it can always be applied to new patterns. The important point is that this process will never be terminated.

So long as the formalization is relatively easy, as is the case in physics, the system of rational metamathematics remains useful in a positive sense, i.e. in it correspondents of the patterns of reality will be sought.

Beginning with a certain stage the entire mystery of life and its apparent irrationalism depends upon the fact that the process of formalizing reality becomes exceedingly difficult and the formalization cannot be achieved by simple application of the method of induction. It is conclusively known, for example, that the patterns of reality employed in law are crude and inept as was shown by Petrażycki.[2]

Because of such limitations intrinsic to the process of formalizing reality, attempts to construct legal systems, known as codes, must be regarded as misdirected. All which is done along these lines must be acknowledged as a sad necessity. The distinctive task of law is the discussion of such conceptual patterns as crime, property, responsibility, etc. It is known that discussion of these patterns leads to vehement disputes and involves harmful tendencies and stubborn prejudices. Obviously such discussions are completely independent of the problem of truth and falsity. They are connected only with the problem of the meaning of concepts.

[1] Emile Meyerson: *Identität und Wirklichkeit*, deutsch von K. Grelling, Leipzig, 1930, p. 426.

[2] Cf. L. Petrażycki: *Wstęp do nauki prawa i moralności*, Tłum. Landego, (*Introduction to the Science of Law and Morality, translated by Lande*), Warszawa, 1930, p. 21 f.

In other words they involve the question whether the formalization of reality achieved by means of a given concept is adequate.

If, for example, a peasant has been put in chains because he has removed a piece of oak from the woods, and maintains that the oak belongs to no one since God planted it, the resolution of his claim depends upon the decision whether reality should be formalized by means of one or another concept of property, i.e. upon the solution of the problem of formalizing reality in this case. Naïve application of the principle of contradiction is not sufficient in this situation. It cannot be said that the piece of oak either belongs to the landowner or does not belong to him. It is the property of the landowner in the eyes of the landowner; it is not the property of the landowner in the eyes of the peasant. To say, as dialectic requires, that it is and it is not, is to say too little, as has been observed. Such a statement would only confirm that the concept of property is not applicable in this situation. Moreover in such a statement the unquestionable fact would be ignored that two different concepts are employed, i.e. the term is being used in two fundamentally different senses.

If it is recognized that using the same term in different meanings, i.e. operating with different patterns of reality, is unavoidable, the problem becomes very complicated because there are no criteria sufficient to decide which meaning should be used at a given moment. If one individual should decide in favour of one convention, another individual could propose the acceptance of a different convention. At this point the borders of the exact sciences are crossed and the domain of individual decisions begins. However, it by no means follows that at this point complete freedom of choice reigns. On the contrary it is precisely here that very strong inner compulsions are encountered in the face of which the individual is distraught. The problem of metascience is the analysis and classification of these compulsions.

The method of metascience eliminates once and for all the oppositions between subjectivity and objectivity. What is called the subject in science becomes objectified when metascience is applied, i.e. the subject is taken to be as much an object of experience as any other object. The living subject appears only at the actual moment of writing these words. However, it is not possible to speak meaningfully of this subject because it is possible to talk significantly only of objects. Already the philosophers of India knew that silence alone is possible concerning a living subject. Metascience provides a technique for introducing the same kind of distinction, by introducing the metascientific theory of types, which is similar to the metamathematical theory of types.

If anyone speaks about his own thoughts, in other words if he occupies himself with some one system of science, he is engaged in a science of higher type. The fact that he is engaged in this science cannot be investigated in a science of lower type. Thus each system of science necessarily has a problem delimited by the type of the formalization which is given.

For example, if it is asked whether the criteria of reality which have been set up are concerned with such facts as that these very criteria have been set up, the answer is in the negative. For handling such a problem a formalization of higher type is required and the criteria for this second formalization must be so constructed that they automatically are of a higher type.

It may be remarked that by means of this theory all the paradoxes and false proofs of metaphysics old and new (in particular Nelson's critique of epistemology) [1] are automatically eliminated.

2. It is evident that the effort required to construct the formalization appropriate for the system of metascience is great. Obviously the accurate formalizations which have been worked out for the subject matter of mathematics may not be completely accepted because of practical difficulties. It cannot be maintained that practising mathematicians actually carry through their reasonings in accordance with these formalizations. In the earlier chapters of this book, semantical abbreviations were introduced, which were followed by abbreviations of reasoning. Finally, on the basis of the conviction that reasonings can be carried through with complete accuracy in a given case, intuitive reasonings were employed. If a completed system of expressions is at hand, such a procedure is permissible without fear of subsequent obscurity. Once a properly worked out scientific system is in existence, intuitive operation with such a system is permissible. It may then be said not that within the given system certain reasonings can be conducted, but that certain problems can be formulated

[1] Cf. *W. R.*, pp. 38–9.

within the framework of that accurate system. If standards of excellence are supplied it may be claimed that the character of accurate systems are known. A certain amount of license may even be permitted and a system not so perfect as that of rational metamathetics, but satisfying the main requirements demanded by the perfect model may be discussed. In this way it is possible to regain the freedom at first excessively restricted by an extremist mania for precision. It is openly admitted that pedantry is not desired. The intent is only to avoid mistaking the approximate for the precise. On the other hand it is not worth while to take pains with precise proofs where fundamentally only confusion reigns. The task of rebuilding the foundations of science from the very bottom and of establishing a standard of precise formulation has been undertaken to deal with such situations. Once this standard of precision and the gaps through which idealistic illusions enter thought are known, it is possible to proceed more freely, just as the factory worker or railroad employee becomes less and less troubled by pedantic regimentation as he becomes more familiar with the routine of his work. It must be taken into account that such situations arise under present conditions of life, where actual living is swifter than thought. Thought is slow and inept and it lumbers along after life with the greatest difficulty.

Consequently any serious attempt to formalize even a partial system of reality is aimless. The constructive attempt of Professor Carnap [1] would be nothing more than a scholastic plaything, were it not for its polemic significance derived from its defence of the sensationalistic concept of reality. I think that this endeavour has been carried too far. It cannot be taken as definitive, since it depends upon the obsolete, simple theory of types and must be supplemented by a large number of familiar enunciations, borrowed from the store-house of an old philosophic dictionary.

A system which itself requires to be explained is but a fragment. It cannot be the basis of a complete system of reality but only an auxiliary device. This is the only possible status of such a system.

In a recent article Carnap [2] tried to establish his position by means of the following argument of Wittgenstein.

[1] Cf. Rudolf Carnap: *Der logische Aufbau der Welt*, Berlin, 1928.
[2] Cf. Rudolf Carnap: "On the Character of Philosophic Problems," *Philosophy of Science*, vol. i, 1934, pp. 7–8.

Wittgenstein [1] permits the use of propositions given in everyday language although he recognized that they have no clear meaning. Their only purpose is to facilitate the understanding of the propositions of the system of natural science. Understanding can be facilitated by means which are not understood. When the end is attained, the means can then be disregarded.

This conception is very ingenious and is based upon observations worth considering. The explanation given by a swimming teacher may not be entirely clear, but once anyone has learned to swim, he will not worry about the fact that the language used to obtain this result was not very accurate.

However, it must be kept in mind that this analogy would be correct only if it were actually possible to construct a closed system of metascience. But since this is impossible, as has been seen, the unformalized aids to understanding give the impression that they are important while the completely formalized systems seem trivial and of secondary importance. Hence everyday language must ultimately be employed : the formalized systems serve only as a means for controlling the caprices of this language and the illusions flowing from it.

I must confess that systems of symbolic logic, worked out with extreme accuracy but based upon idealistic presuppositions, are much less clear to me than are trivial descriptions in which everyday language is employed. It must be kept in mind that the endeavour to attain precision is a two-edged weapon. It may deprive the individual of his power to react automatically which is associated with everyday language and does not always replace it by an intelligible equivalent.

I think that the following quotation from Professor Dupréel is decisive here :

> "The formuli of the scholar are meaningful and of interest only in so far as they are related to certain ordinary and earlier representations for which they may be substituted in conformity with certain conventions." [2]

3. The elimination of the subject leads to *materialism*. However, it must be kept in mind that the concept materialism is but crudely defined. The term itself can have many meanings.

[1] Cf. Ludwig-Wittgenstein : *Tractatus Logico-Philosophicus*, London, 1922, p. 189, p. 77.

[2] Eugène Dupréel : *Traité de morale*, Travaux de la Faculté de philosophie et des lettres de l'Université de Bruxelles, t. IV, Bruxelles, 1932, p. 689 n.

What is relevant here is that the thing designated as an object of experience or matter is not itself apprehended. Only patterns which impose themselves on an individual are treated. These patterns may differ fundamentally. Things and their properties, the properties of an unapprehended material process, the so-called sensory elements [1] or finally the intuitively represented elements or *images* are such patterns.[2]

At a given moment each of these may seem to be the only true reality although at a subsequent moment it may be regarded as an inept fiction and be replaced by something else. The common sense notion of some *true* reality given immediately to the mind in various ways, mentioned by Bergson,[3] must lead to an idealistic fiction and the irrationalism connected with it. The difficulties which he points out [4] concerning this view cannot be avoided because this *true* reality is itself only one of the possible realities. At some moment it disappears and is replaced by something which previously was regarded as but one aspect of it.

There is thus no basis for speaking about one *true* reality as a totality of determined objects. It must be insisted that there are many concepts of reality. At different times different concepts must be employed to describe the immediate experience had when it is declared that something is real. I once called such an experience *metaphysical* [5] because I wished to point out that they lead to metaphysical illusion. From the metascientific point of view they are *objective* because they are independent of and accessible to all people who wish to consider these matters sufficiently carefully. But at the same time these experiences contain subjective elements. They seem absolutely true even when it is known that after a certain length of time they can be replaced by other experiences.

Consequently the criteria of reality play a secondary role in comparison with the criteria of sound reason. If the former are taken as the basis for constructing a world view, contradictions inevitably arise. But if the criteria of sound reason are employed and if in addition use is made of the

[1] Cf. Ernst Mach · *The Analysis of Sensations*, translated by C. M Williams, Chicago, 1914, pp. 2–6.

[2] Cf Henri Bergson: *An Introduction to Metaphysics*, authorized translation by T. E. Hulme, New York, 1912.

[3] Bergson: *l c.*, p 65.

[4] Bergson: *l.c*, p. 67.

[5] Cf. *Z K. D. P.*, p. 97.

metascience based upon them, serious difficulties can be overcome. In this latter case it is no longer necessary to dismiss what is regarded as an inner compulsion, and all efforts may then be devoted to clarifying the confusions to which allusions have been made.

Clearly even the most extreme individualism may be useful, provided that it is consciously employed and that objectivity is not at this time either naïvely or cynically claimed.

4. The strength of the concept of natural reality lies in the fact that the criteria of sound reason, which are employed, have developed simultaneously with the criteria of this reality.

It is not possible to avoid entirely a natural view of the world. Its effect is so strong that no one can escape a more or less conscious return to it at certain periods of his life. Even if one is convinced that natural reality is but a certain kind of myth inherited from uncritical periods of past centuries frequently it is not possible to avoid the conviction that everything else is an artificially produced theory. It is also felt that reality is one and is precisely such that the persons and things which constitute it have properties of colour, sound, taste, etc., independent of the individual percipient and that these properties are intrinsically pleasant or unpleasant, good or bad, beautiful or ugly.[1]

The theory of light waves, the physical analysis of the play of colours and forms in the world about the individual are of no aid in avoiding a natural view of the world. An individual knows that this dress is blue, that this table is rectangular, and all changes which appear at the moment of observation are held to be of secondary importance. It is of no avail to recall that the continued use of this approach subjects the individual to all those prejudices of the popular view of the world which arouse disdain and contempt. The procedures relevant to the concept of natural reality overcome all obstacles; they generate in the individual an inner need to identify the criteria of natural reality with those of sound reason.

The weak point of the concept of natural reality concerns the way in which visual images are treated. These images, as is known, can be very clear, with the result that they do not differ intrinsically from things and persons. Uncritical acceptance of this situation leads to dangerous consequences.

The confusions which rise from uncontrolled visions of

[1] Cf. W. Heinrich: *Teorje i wyniki badań psychologicznych* (*The Theories and Results of Psychological Investigations*), Warszawa, 1902, p. 4.

natural reality, such as are involved in all mythology, have proved very dangerous.

Consider the following example:

In Constantinople a certain man and his son were condemned to death because a bishop had a revelation that this man had thwarted the performance of a miracle by magic practices.[1]

St. Thomas Aquinas writes:

> "Therefore it must be admitted that all the transformations of corporeal things which can be produced by certain natural powers, to which we must assign the seeds above mentioned, can alike be produced by the operations of the demons, by the employment of those seeds; such as the transformation of certain things into serpents or frogs, which can be produced by putrefaction."[2]

The eminent Sprenger, the author of *Malleus Maleficarum*, believed that the devil can change a woman into a cat at any time.[3]

The famous work of Brother Ubaldo Stoiber, entitled *Armamentarium Ecclesiasticum*, was published in Augsburg in 1726. In this work the author gave precise indications how to distinguish the devil from an angel, from a soul in purgatory, and from a magic phenomenon. Among other things he seriously considered whether a soul in purgatory can assume the form of an animal and concluded that this possibility is not precluded, since a certain Franciscan monk saw a naked woman in the company of two wolves. The woman turned out to be the soul of a certain libertine and the wolves the souls of two priests who had been her confessors.[4]

To-day all this nonsense is known to have been caused by superficial observation of such phenomena as sense-illusions. More precisely the reality of images which was substituted for natural reality was interpreted by means of the criteria of the latter.

[1] Cf. Charles Louis Montesquieu: *The Spirit of Laws*, translated by Thomas Nugent, Cincinnati, 1873, Bk XII, ch. 5, p. 216.

[2] St. Thomas Aquinas: *Summa Theologica*, part i, translated by the Fathers of the English Dominican Province, New York, Cincinnati, Chicago, 1912, Part I, Third Number, vol iii, p. 503

[3] Cf. Jules Michelet: *La sorcière*, Paris, 1862, pp. 182, 198.

[4] Cf. Brother Ubaldo Stoiber. *Armamentarium Ecclesiasticum*, Augsburg, 1726, Pars II, p. 196. Professor Podlacha observed in conversation with me that this entire vision was obtained from the old Christian iconography. (Thomas Garter: *The most virtuous and godly Susanna*, 1578, London, 1937.)

The use of the patterns of natural reality tend to support reaction because they have been sanctioned by tradition, crystallized in everyday language, and because that which is familiar differs from that which is a recent innovation. For the latter is never perfectly clear and is always uncertain.

There is no basis for the assertion that it is possible to avoid the return to the concept of natural reality at certain times, particularly when life has become very pleasant or very difficult. However, it is sufficient merely to be aware of this possibility in order to prevent the occurrence of its pernicious consequences. The attempt to eliminate completely the concept of natural reality, leads to the contraction of the extent and bounds of experience ; while on the other hand reinstitution of this concept may even prove useful if accompanied by a sufficiently critical spirit. The only point that must be born in mind is that this *concept of reality* is *of higher type* and becomes dominant despite the knowledge that other concepts of reality may impose themselves as exclusively valid.

5. Profound researches in the natural sciences, increasing familiarity with the telescopic and microscopic worlds, together with the endeavour to discover the relations and laws governing these worlds can lead an individual to a completely different concept of reality. It is of secondary importance that the concept of matter undergoes continual evolution and has not received a definition which is precise and consistent with experience. What matter is and what forces act upon it make no difference. Whatever matter may be it is impossible to evade the feeling of absolute certainty that matter is the only true reality, that mental life merely indicates neural processes, and that in general psychic states are secondary phenomena, without influence upon the history of the universe. The reality dealt with in this case I call physical reality. As in the case of natural reality, for some people it is the only reality and no one can be sure that at a given moment it does not obtrude upon him as the exclusively valid concept of reality. Once again the knowledge that at other moments of his life other concepts of reality impose themselves upon him with equal force is of no avail. The power of suggestion is too great. He has but a pitying smile for other inner compulsions since he takes it for granted that they lead to illusion.

Eddington offers an interesting description of the vacillation between the concepts of natural and physical reality. He speaks of two tables, the usual table and the scientific table, i.e. a certain complex of atoms.[1] This account indicates that the theory of the plurality of realities, which fifteen years ago professional philosophers regarded as absurd, has begun to pervade the minds of scholars.

Physical reality originated in the sixteenth century with the development of mathematics, the natural sciences and the theory of perspective (Leonardo da Vinci). It formed the basis for the great philosophical systems of the seventeenth century. Hobbes and Locke as well as Descartes, Spinoza, and Leibniz based their views upon this conception of reality, although they conceived its relation to an abstract conception of God and the soul in different ways. Some of these writers have been classified as materialists, others as spiritualists, but such a classification is of purely theoretical significance; for what is relevant is not the criteria of reality but their interpretation.

At all events it must be taken as well established, in so far as anything can be well established in such matters, that these thinkers confine themselves to employing concepts of reality associated with the material processes and the laws governing them.

This tendency appears most strongly in the system of Spinoza. His geometrical proofs are paralogisms, but his direct intuition of the independent existence of a Nature-God imposes itself with ineluctable force.

The absolute determinism of Nature and the very petty role of man in the universe has the character of an absolute truth concerning which all polemic is futile.

> " A thing which is determined for the performing of anything was so determined necessarily by God and a thing which is not determined by God cannot determine of itself to do anything.
>
> " A thing which is determined by God for the performing of anything cannot render itself undetermined.[2]
>
> " To act from reason is nothing else than to do those things which follow from the necessity of our nature considered in itself." [3]

In time these thoughts came to be regarded as elementary truths which were substituted for the popular view of the world. They are influential to this very day.

[1] Sir Arthur Eddington: *The Nature of the Physical World*, Cambridge, 1929, pp. xi–xiv.

[2] Baruch Spinoza: *Ethics*, Everyman edition, New York, 1910, Part I, Props XXVI and XXVII, pp. 21–2.

[3] Spinoza. *l.c.*, Part IV, Prop LIX, Proof, p. 181.

Spinoza became the father of contemporary materialism, whose essential difference from Spinozism lies in the fact that in it the concept of God is replaced by the concept of Nature.

La Mettrie took an important step forward in the construction of the concept of physical reality.[1] He was a physician. Self-observation, while in a feverish condition, led him to the conclusion that psychic processes can be explained by appealing to physiological ones. This was a fruitful idea and determined the further development of the study of man.

The materialistic conceptions of the eighteenth century were wrecked upon the concept of the subject, which could not be successfully handled with the meagre logical apparatus of that time. Consequently the concept of physical reality was submerged for a long time on the confusions of Hegelian irrationalism.

The chief influence of the conception of Hegel has been to support the view that because physical reality continually changes, it cannot be accurately described by means of concepts. This idea has been accepted by Marx, Engels, Lenin, and their successors. It has been confirmed by the results of physical experiments. Everything seems to indicate that the concepts with which physics operates are only very primitive formalizations of reality. However, it by no means follows that the dialectical method gives knowledge which is any more accurate. Nevertheless it should be emphasized that physicists are invariably carried away by the naïve hope that they have already discovered the essence of matter and when this hope proves deceptive, in their despair they betake themselves to another concept of reality. But it is obvious that merely because a certain description of matter has proved to be inaccurate it cannot be concluded that there is no matter.

The concept of physical reality arises from criticism of the concept of natural reality. It is therefore a concept of reality of higher type. Neglect of this phenomenon leads to serious misunderstandings. For example, some philosophers forget that because in physical reality events are determined by material processes, it by no means follows that there is no free will in the ordinary sense. The fact is that human beings have intuitive criteria which serve to distinguish voluntary and compulsory activities and which permit them to adjust themselves in a rough way in their everyday life.

[1] Cf Überweg-Heinze: *Grundriss der Geschichte der Philosophie*, 9. Aufl, Berlin, 1901, Bd. 3, p. 244.

These criteria fail in more complicated cases and over-estimation of their value is an obvious error. Nevertheless it would be wrong not to take account of them. Such an error led to the confusion of concepts which has resulted in the complete denial of the possibility of ethics.[1] Thus the problems of ethics are independent of the problems raised in connection with the material processes which determine the lives of men; they have nothing in common with the so-called problems of *responsibility*. It may be agreed that men's actions are dictated by their basest instincts, and it can even be conceded to Professor Freud that a mother's love for her son is nothing but a form of sexual egoism.[2] But none the less mothers who show concern for their children will have to be distinguished from those who do not; the former will be called good, the latter bad. To abandon this classification would only fruitlessly impoverish life. It must, however, be kept in mind that uncritical use of this classification may lead to injustice.

6. The foundations of the concept of the reality of sensations or of sensory elements were erected by David Hume.[3] Ernst Mach later gave a clearer formulation of this position. Avenarius should also be mentioned in connection with Mach. Avenarius [4] created the conception of pure experience, i.e. experience from which all conjecture, habit, and convention have been eliminated. He called his doctrine *empirio-criticism.*

The concept of pure experience is not sufficiently clear. Mach's formulation was much more lucid. He wrote as follows:

> "Colors, sounds, temperature, pressures, spaces, times and so forth are connected with one another in manifold ways; and with them are associated dispositions of mind, feelings, and volitions. Out of this fabric, that which is relatively more fixed and permanent stands prominently forth, engraves itself on the memory and expresses itself in language. Relatively greater permanency is exhibited first by certain complexes of sounds, colors, pressures, and so forth, functionally connected in time

[1] Cf. Otto Neurath: "Soziologie im Physikalismus," *Erkenntnis*, Bd 2, 1931, p. 418.

[2] Cf. Sigmund Freud: *Civilization and its Discontents*, authorized translation by Joan Riviere, New York, 1930, p. 89.

[3] David Hume: *An Inquiry concerning Human Understanding*, Chicago, 1930, esp. Sections II and III.

[4] Richard Avenarius: *Kritik der reinen Erfahrung*, Leipzig, Bd. I, 1888, Bd. 2, 1890.

and space which therefore receive special names and are called bodies. . . .

"Further that complex of memories, moods, and feelings, joined to a particular body (the human body) which is called the 'I' or 'Ego', manifests itself as relatively permanent.[1]

"The visible, the audible, the tangible are separated from bodies. . . . The complexes are disintegrated into elements, that is to say, into their ultimate constituents, which hitherto we have been unable to subdivide any further." [2]

It should be noted that this characteristic view of Mach has nothing in common with classical idealism. Sensations are indeed elements of reality, but they are held to be independent of the problems raised in connection with the conception of the external world and the so-called "Ego". On this view both the external world and the so-called "Ego" are simply complexes of sensations. Instead of talking about sensations it is permissible to speak of *sensory* elements. The independence of this concept of reality from the subject is thus forcibly emphasized.

Persons and things have a place in the concept of the reality of sensory elements but they are conceived differently than in the concept of natural reality. The sun is a luminous disk. The moon is an object which changes its shape weekly. Sometimes it is round, other times it has the shape of a sickle. The concepts of the sun and the moon conceived as ellipsoidal material masses are but artifacts created by the individual for the purpose of adjustment in the concept of the reality of sensations. Similar remarks may be made with regard to the concept of the human organism and of the physiological processes occurring in it.

The problem of science is to discover constant relations between those artifacts.

The phenomenon of the duplication of a lead pencil, when observed closely, can confirm the obtrusion of the concept of the reality of sensations. Both pencils appear equally real. The thing cannot be duplicated. Therefore no thing presents itself in accordance with the concept of natural reality. What is called a thing is an artifact. The perceived pencil is merely an image.

Obviously this experience can be disregarded and attributed

[1] Ernst Mach: *The Analysis of Sensations*, translated by C. M. Williams, Chicago, 1914, pp. 2–3.

[2] Ernst Mach: *l.c.*, pp. 5–6.

to *anormal* conditions of observation, but with equal right the conception of normal conditions of observation might be regarded as a very confused artifact and it would not be regarded as the basic criterion of reality.

Bertrand Russell explains the emergence of the concept of the reality of sensations in connection with an analysis of a table analogous to Eddington's discussion of the table. In the language of Eddington, it would be table number three. This table does not have the characteristic shapes or colours of a table, but is simply a complex of sensations which varies with the point of observation, the conditions of illumination, or the position of the sense organs.[1]

The identification of various complexes of sensations as one table involves a certain convention. A similar convention is also involved in the concept of physical reality. In each case the impermanence of material complexes must be taken into account.

The concept of the reality of sensations has had great influence upon the development of contemporary physics. It has heightened criticism considerably and overcome idealistic illusions. Nevertheless a great danger is involved in this concept because of the tendency not to take into account the possibility of fundamental alterations in the extent and bounds of experience. It is worth noting that such tendencies appear at the very moment when man has gone much further beyond the bounds of experience of tradition than the most daring visionaries of the past could have hoped to go. It is true that the extension of the bounds of experience can never be achieved by means of existing patterns; consequently this possibility could be termed a fiction. But such a fiction is not synonymous with nonsense as the adherents of the concept of the reality of sensations would have us believe. The concept of the reality of sensations when combined with rejection of fictions dooms us to move in a closed circle of experimental data and inevitably leads to unusual restriction of man's creative possibilities.

The concept of the reality of sensations assumes such a form in a recent pronouncement of Professor Heisenberg, who has already been mentioned. Heisenberg believes [2] that the hope

[1] Bertrand Russell. *The Problems of Philosophy*, New York, Chs. 1 and 2, pp. 42–3, 73–4.

[2] Cf. W. Heisenberg: "Wandlungen der Grundlagen der exakten Naturwissenschaft in jungster Zeit," *Die Naturwissenschaften*, Bd. 22, 1934, Heft 40, p. 674.

of enlarging the extent and bounds of experience without limit is as vain as the dreams of former investigators concerning the discovery of the ends of the earth. I do not think this simile is a happy one. It would be more appropriate to say that belief in fixed limits to experience is similar to the old belief in the fixed limits of the earth.

In social affairs the concept of the reality of sensations leads only too often to cynical egoism of the type exemplified in Aristippus [1] and even to calculated ruthlessness. It is seldom accompanied by the social instinct and the inclination to devotion. Lenin emphasized these dangers with unprecedented acuteness.[2]

Lenin was able to refute Mach, Avenarius, and the related immanent philosophers, but he was not able to overcome in his own person the reality of sensory elements and neither could Emile Meyerson [3] nor his intellectual successor Dr. Metallman.[4]

The concept of the reality of sensory elements cannot be refuted and some day perhaps philosophers will become reconciled to this fact.

A particularly vicious example of a one-sided concept of the reality of sensations is *behaviourism*, the doctrine in which the psychic states of others are rejected as intrinsically unknowable. But one's own past and future psychic states are equally inaccessible to direct experience, as Professor Łapszin rightly remarked.[5]

If an individual takes his past and future sufferings into account he has no right to reject the sufferings of others.

In order to characterize the behaviouristic doctrine whose growth I regard as one of the most serious dangers threatening us at present, I cite the following paragraph [6]:

> "In this case the inquisitors having seen the obstinacy of the accused order that he be subjected to torture. . . .

[1] Cf., *e.g.*, Überweg-Heinze: *l.c.*, Bd. 1, p. 145.

[2] Cf. Nikolai Lenin: *Materialism and Empirio-criticism*, translated by David Kvitko, New York, 1927, vol. 13 of *Collected Works of V. I. Lenin*.

[3] Cf. Emile Meyerson: *Identity and Reality*, authorized translation by Kate Loewenberg, New York, 1930, pp. 293, 298.

[4] Cf. J. Metallman: *Determinizm nauk przyrodniczych* (*The Determinism of the Natural Sciences*), Krakow, 1934.

[5] Cf. I. I. Łapszin: *Oprowerżenie solipsizma, Uczenia zapiski* (*Scorn of Solipsism, Lecture Notes*), Praga, 1924, p. 18.

[6] Cf. Eduard Quinet: *L'Ultramontanisme ou l'église romaine et la société moderne, Oeuvres Complètes*, Paris, 1844, 3rd ed., vol. 2, pp. 196–7, n. 1.

Quinet quotes here from *Arsenal sacré ou Pratique de l'office de la sainte inquisition imprimé à Rome 1730 et dedié au glorieux inquisiteur Saint Pierre*, p. 263.

"And they order that the accused be led to the place of torture, that he be bound and fastened to a rope.

"Led in this way, while being undressed, bound, fastened to the rope he is graciously induced, paternally admonished—*benigne monitus, paterne exhortatus*—by the inquisitors to speak the truth and not to wait until he is lifted by the rope as he will be if he persists.

"Then the inquisitors, seated and seeing that the accused, who has been undressed, bound and fastened to the rope, refuses to speak the truth order that he be hung.

"The prisoner being lifted, begins to shriek, saying: 'Alas! Oh, Saint Mary, etc. . . .' or else he remains silent."

This description is in many respects like the medical reports of experiments performed on *charity patients*, the poor, defenceless people. Reports of vivisections and of psychological experiments performed on animals are of the same character.

The abominable sterilization law, obligatory in Germany, belongs to the same category.

7. The concept of the reality of intuitively represented elements appears in its pristine form among children and primitive peoples because they confuse the criteria of the dream and waking life. This concept of reality appears more immediate than the concept of natural reality. The concept of natural reality is arrived at in the course of the struggle which has lasted many centuries between the criteria of sound reason and the criteria of intuitively represented reality.

An analysis of experience based upon sound reason leads the individual to divide the objects surrounding him into two fundamentally different categories. The first contains people and things, and events which occur in the world of people and things.

Failure to acknowledge these obvious facts is the outcome of confusing the criteria of reality with the criteria of utility. Thus St. Augustine maintained that food seen in a dream is less real than the food of the waking life because the former is not nutritious, while the latter is.[1] However, he overlooked the fact that poison swallowed during waking life would, on this criteria, be even less real, because it not only fails to prolong life but actually shortens it. Neglect of the role of images is motivated by the fact that only by disregarding

[1] Cf. *Confessions* of St. Augustine, translated by Dr. E. B. Pusey, London, 1907, Bk. 3 (vi) 10, p. 38.

their appeal can their dangerous growth be avoided. The Witches Sabbath was at first considered a diversion, unseemly but amusing and cheap. Persecution made it a forbidden delight and much to be desired.[1] The disappearance of the Witches Sabbath is to be attributed to changed conditions of daily life on the one hand, and the increase of tolerance on the other. The condemnation of free love was effected even more easily, although the condemnation of amours with the devil required effort. Journeys on broomsticks have gone out of fashion.[2]

The struggle with naïve forms of the concept of the reality of images is no less a social necessity than is the struggle with all manifestations of exaggerated individuality. However, the situation is different the moment a concept of the reality of intuitively represented elements of higher types is concerned, i.e. the concept of reality obtained by profound researches on the problem of reality. I have in mind an experience of the type about which Descartes wrote in the first of his *Meditations*.[3] Descartes was concerned with the view that there is no clear cut method for distinguishing the waking from the sleeping state. But he was appalled by what at first sight seemed the alleged necessity of this conclusion and felt this to be an unendurable situation. He saw in God the only possible way of escape and sought shelter in Him as does a frightened child in the bosom of its mother. It seems to me, however, that if the waking state could not be clearly distinguished from sleep, we would not be entitled to trust our reason and would be doomed to eternal ignorance.

Descartes forgot that the waking state can be differentiated from sleep only because of the greater degree of coherence of the former. This argument is due to d'Alembert.[4] If Kant's philosophy is examined from this point of view it can be regarded as formulating the system of laws of the reality of images. Such an interpretation of Kant's philosophy obviously eliminates the *thing in itself*. But the Kantian concept of the thing in itself has *from the beginning* been permeated by

[1] Cf. Michelet: *l.c*, p. 142.

[2] Cf. Dr. Erazm Majewski: *O wpływie nauk przyrodniczych i lekarskich na pokanie przesądów ciemnoty i zniknięcie chorób umysłowych, zwanych demoniczemi. (Concerning the influence of the Natural Sciences and Physicians on the Overcoming of Ignorant Prejudices and the Disappearance of Mental Diseases Called Demoniac)*, Kraków, 1882, p 40

[3] Cf. Renée Descartes: *Selections*, edited by Ralph M. Eaton, New York, 1927, pp. 90–5.

[4] Cf. Łapszin: *l.c.*, p. 22.

ambiguity.[1] It has been as contradictory a term as *Polish Emperor*, for if something affects an individual in a specific way so that his behaviour can be predicted in advance, it cannot be asserted that that thing is unknowable. (Argument of Plekhanov.) [2]

If the thing in itself is rejected, then the image is the only reality. Persons then are simply images. Similarly the sensations of an individual are images. As Bergson [3] rightly remarked, man has no pure sensations. The view introduced by Hume [4] that images are weak reflections of sensory elements cannot be maintained. On the contrary sensations depend primarily upon images and a sensation can be regarded as a particular kind of image. To become convinced of this, it is sufficient to note the increase in one's power of observation under the stimulus of hints and reminders supplied by others. It suffices to point out that in observing something new under unfamiliar circumstances, e.g. by means of a microscope, what is to be done is not known until it is known what to look for.[5]

The concept of the reality of intuitively represented elements of higher type plays an important role in creative activity. Consequently the common opinion that it is difficult to draw a sharp line of demarcation between the genius and the insane person is sound. Sound reason is decisive here. In so far as it is observed that the criteria of reality lose their significance when they are in opposition with sound reason, the reality of images, to use the expression of Engels,[6] is a *thing for us*. But whenever images are emancipated from the criteria of sound reason, disintegration and chaos set in. The institution of sound reason and much philosophic culture was needed in order to prevent the undesirable dominance of the concept of the reality of images. To be able to appraise the concept of the reality of images the claim which the concepts of the reality of sensations and of physical elements can make upon one must have been experienced and it must have been fully realized that there is no such thing as a *true* conception of

[1] Cf. Cassirer: *Kants Leben und Lehre*, Berlin, 1921, p. 231.

[2] Cf A. Deborin: *Wwedenie w filosofiu dialekticzeskogo materializma* (*Introduction to the Philosophy of Dialectical Materialism*), Moskwa, 1931, p 177.

[3] Henri Bergson *Matter and Memory*, authorized translation by Nancy M. Paul and W. Scott Palmer, London, 1911, p. 24.

[4] David Hume *l c.*, pp. 14–15.

[5] Cf. the interesting article of Dr. Fleck: " Zur Krise der ' Wirklichkeit '," *Die Naturwissenschaften*, Bd. 17, 1929, Heft 23, p 425.

[6] Cf. Lenin: *l.c.*, pp. 75–7.

reality. Unless this realization is attained the concept of the reality of images is an idealistic fiction which sooner or later will collide with sound reason.

Idealism is a false appraisal of the concept of the reality of images. It springs from neglect of the fact that within this framework of reality, truth, and falsity cannot be differentiated. Thoroughly false as well as true assertions may always fascinate an individual and may seem to him an inexorable necessity. As long as it is not observed that the criteria of truth and falsity are independent of the concept of reality which is dominant in experience, the individual will be submerged in idealistic error. But to reach this conclusion greater familiarity with the concept of the reality of images is necessary because it provides material on the basis of which comparisons might be made. The struggle with idealism conducted in the name of the concept of the reality of images is the sequel of the struggle against Pharisees. In other words it is like a life full of vitality contrasted to a life empty of vital interest.

8. The theory of plural realities is a relativistic doctrine. However, it has nothing in common with conventionalism. Because of the confusion of these two concepts relativitism is held to be socially dangerous. *Absolute obligation, absolute good, and absolutely inelastic police orders* are regarded as the basis of a wise social organization.

The conventionalistic position will now be examined.

The history of the customs of all times and countries shows that the individual seldom applies absolute criteria to himself and applies them to other people. It seems to me, however, that the very people who apply these criteria most consistently to themselves are the most dangerous socially because they believe that being without sin, they have the right to cast stones at others.

The interpretation of relativism as an opportunistic doctrine according to which that which is of greatest advantage at the given moment is pursued depends on confusing strong belief with a verbal formulation.

Because alternative possibilities are recognized it does not follow that it is necessary for the individual to swerve from his chosen path. If one regards his way as the only possible way, he risks danger of error because some deviation is an inevitable necessity.

The criteria of sound reason must be employed to unmask invalid arguments arising from a desire for absolute truth.

It should be added that this desire frequently springs from the inadequacy of the little knowledge the individual does have. In brief he knows nothing, since it is impossible to know absolute truth and it is not worth while to know relative truths. The system of metascience is necessary to strengthen sound reason precisely at those points where its criteria cease to operate directly. Sound reason is not sufficiently courageous and does not differ from common sense sufficiently to pursue its own course against authority, idealistic logic, and public opinion. In this struggle sound reason requires the aid of an external logical apparatus which although it is its own product, functions independently of its whims. At moments of fatigue faith in the criteria of sound reason may be lost. At that time, however, a system of metascience which functions automatically continues to operate. This obviously does not settle the matter definitively since it is also possible to lose faith in this system. However, if it must be decided whether a machine-gun or one's fists should be employed, it is obviously preferable to defend oneself with the gun or at least to have the gun in reserve.

The distinction between the events of ordinary life on the one hand and unusual experiences and mysterious events on the other cannot be drawn precisely. There is no province of daily life into which some bewildering perplexity or reckless frenzy may not steal. Even the simple problem of a bookkeeper adding columns of figures may involve what seems like a diabolical caprice which is completely at variance with law. That certainly is his reaction when the answer does not come out right and he has been unable to make the calculations agree. The question could therefore be raised whether the calculations could ever be made to agree or whether some occult power thwarts his designs. If an affirmative answer is given to the second alternative he will be transported into a naïve reality dominated by personal experience, images, and visions which lead him far beyond daily life.

In general, however, he will not succumb to these unusual personal experiences. Acquaintances are appealed to who usually can lead him out of such situations. If they fail they regard him as insane and put him in a sanatorium. In such an event he obviously will not believe that he has been justly treated. On the contrary, he will be convinced that the concept of reality they employ in putting him out of the way is not worthy of serious consideration. To the argument that it is not possible to remain alive for long if daily life is disregarded, he could simply retort: Why should one remain alive? It is far better really to have enjoyed one brief glimpse of true reality than to have lived in unenlightened error for a long time.

Although there is no way of convincing the overtired book-keeper that the multiplication table is correct, there are no good reasons to question it.

This situation is disturbing only if the criteria of sound reason are regarded as the source of absolute truth. But if sound reason is taken to be only an island of safety, a refuge from an arbitrarily created fiction which claims to be the truth (the making of truth) [1] it is possible to console oneself with the thought that a night full of mystery, error, and fright will be succeeded by daylight bringing with it the return of sound reason. Should the night never pass and if all mankind were to be in the same plight, all the present efforts would be wasted. But the same consequence would follow, if for example some great cosmic catastrophe would occur. To hinder the return of sound reason by holding on to the belief in the existence of ideas and absolute truth is like singing lullabies or telling a bad child fairy tales.

It may be objected that perhaps the singing of such lullabies and the telling of fairy tales provide the only remedy for the situation since according to the theories of James [2] and Schiller that turns out to be the case.

Some have protested loudly against the tedium and intolerable constraint of the criteria of sound reason. In contrasting Aristotle with Plato, Rudolf Eucken wrote:

> "With this disappearance of religion falls away the sincerity of spirit and the universal greatness of thought of Plato. Life receives narrower bounds and its emotional tone becomes more sober." [3]

On the other hand Aristotle himself may be regarded as supplying a sort of narcotic in contrast to the fruits yielded by sound reason and the logic based upon it.

A comparison of the simple and at times naïve truths of

[1] Cf. F. C. S. Schiller: *Studies in Humanism*, London, 1907, pp. 179–203.
[2] Cf. William James: *Pragmatism*, New York, 1907.
[3] Rudolf Eucken: *Die Lebenanschauungen grossen der Denker*, 2 Aufl., Leipzig, 1897, p. 61.

Hume or Mach with the flights of phantasy of Plato, Plotinus, Origen, Spinoza, Hegel, Schelling, Schopenhauer, Nietzsche, or Lev Shestov leaves the impression that instead of the hurdy-gurdy the silver horns of the archangels are playing and leads one to laugh at the unequal competition.

It might, however, be asked whether the tunes of the hurdy-gurdy are not better than the false pathos of operatic mutes disguised as archangels. What guarantee is there that the flights of phantasy of geniuses are not similar to the performances of the *Ring of the Nibellungen* in which second-rate actors with shrieking voices appear instead of real heroes and in which paper maché trees squint ironically at the puffing director of the orchestra. In spite of all the display, Wagnerian music is an unintelligible noise to the public and its power is already spent and is in part banal for those few who understand it. I borrowed this illustration from Leo Tolstoi; but I must confess that to understand it I required great effort because its suggestion of lofty pathos reacted upon me with great force. Works of art determine the atmosphere of intellectual society. They have many other functions for they make life interesting and alluring. However, they frequently lead to grievous excesses and when their immediate appeal ceases distaste and aversion will remain.

This state of affairs is a relic from more primitive days, when moral and artistic criteria were mingled with the criteria of truth, thus producing the confused mixture of convictions which is called common sense.

To this it may be objected that it is better to succumb to sublime illusions even if bitter disappointments may be the consequence, than to be satisfied from the very outset with prosaic essentials and so to condemn life to a closed circle of simple and banal thoughts. After what has been said previously it is easy to deal with this objection.

The concept of reality which dominates an individual's life is not to be judged primarily in terms of the question of truth. The wealth of the inner life of an individual does not prevent the application of the criteria of reasoning. For although the construction of the foundations of science does not rely upon the intervention of God, it does not follow that the yearning for a higher being who understands man completely must be abandoned. However, it must be noted that no consequences can be derived from the fact that this yearning is motivated by the hope for immortality. To restrict oneself to the criteria of sound reason is by no means synonymous with giving up vision and fancy. It is possible to enter into a world of very dangerous fancies without fear of inner confusion precisely because one has clear criteria of analysis. The derivation of social consequences from these fancies and visions will not be sought simply because such criteria are recognized.

Such hypotheses as infinitesimals, the ether, or the actual infinite of Cantor had their source in phantasy unregulated by the principles of sound reason ; their creative role depended upon the fact that what was really of value in them could be absorbed in a system of science based upon sound reason.

No one would seriously propose a mechanization of life and science in such a way that creative imagination and fantastic and ominous thoughts would be completely eliminated ; the forebodings of some philosophers are indeed pitiable. But even more ludicrous is the fear that prejudices will disappear so that life will lack colour and be converted into a sheer vegetable existence. Prejudices are never lacking, for from the ashes of old prejudices arise in our times much more menacing and at the same time amazingly fantastic ones. Likewise there need be no fear that the mechanization of life at any time will advance too far. The alleged mechanization of life in our times is a typical illusion of perspective. I am convinced that a postal clerk in a small provincial town to-day has a much more varied life than the nobility of former times, who spent their days as uninterestingly as the stones in their castles. Even more noteworthy is the fact that they have greater exuberance of temperament and sometimes equally savage passions. It is most amusing that the most ardent defenders of individualism are people of whom it can be predicted in advance what they will be doing in the course of any day during the next few months. Many such examples could be given. I will only add that in the same category should be included the fear that foods will completely disappear when they are being destroyed in bulk, and that the great majority of people will suffer from hunger.

This misunderstanding arises because thorough criticism based upon sound reason is lacking and is dictated by that malicious force slumbering within the human organism which Nietzsche called the " spirit of gravity ".[1]

[1] Friedrich Nietzsche: *Thus Spake Zarathustra*, translated by Thomas Common, Modern Library Edition, New York, Third Part, Ch. LV, pp. 198–202.

It is certain, for example, that if all mankind were castrated, as the eugenics societies seem to wish, very disturbing changes would occur. But it must be explicitly noted that all such thoughts are the product of irrational metaphysics and lack of criticism. To save mankind from such extravagances of over-refined intellectualism a defence of sound reason and the logic based upon it is necessary. For this reason a merciless criticism of prejudices, conducted consistently and stubbornly and using all the means at our disposal, is necessary. I think that the worthiness of this aim will repay the trouble taken a hundredfold.

I cannot hide that the labour involved is very great and its possible results may be very small. But such is the fate always of every great human endeavour. Woe be to him who does not wish to reconcile himself to this great law of life.

APPENDIX

THE FUNDAMENTAL SYSTEM OF SEMANTICS[1]

The general account of Chwistek's system presented in the Introduction does not explain the elaborate technique which he has introduced for the reduction of mathematics to semantics. This technique is not only very difficult but is relatively unfamiliar even to readers well-versed in recent logical literature. For this reason it seems worth while to present a critical account of Chwistek's system,[2] in order to forestall some of the difficulties the reader is likely to encounter in reading the technical portions of *The Limits of Science*. The attempt will be made to offer an explanation of Chwistek's views, which is consistent with his own intent and at the same time takes into consideration some of the interpretations made by other writers.

Chwistek's entire construction depends upon the acceptance of what he calls " intuitive " logic (165), to which the rules of semantics are themselves subject. His use of the adjective " intuitive " seems to have nothing in common with any of the usual interpretations of this word.[3] What he has in mind is a set of three simple forms for the rules of semantics [4] which contain a minimum of everyday language, and two simple rules of reasoning [5] to be applied to sentences obtained from these forms. Such a logic is " intuitive " in the sense that it is never subjected to further analysis but is accepted uncritically and without question as conforming to our usual habits of thought.

[1] Familiarity with Section II of the Introduction, which presents a general picture of Chwistek's aims and methods in constructing his system of semantics, will provide the reader with the background necessary for understanding what follows.

[2] While in the main this account is developed in such a way as to parallel Chwistek's own account, it must be regarded as a supplement to and not a substitute for Chwistek's presentation of his views.

[3] For example Chwistek rejects the dependence of mathematics upon " intuition " in the Kantian and constructivist sense of the term (i.e. sensuous intuition of space or time).

[4] i.e. " **E** *is an expression* ", " **E** *is a theorem* ", and " If **X**, then **Y** ".

[5] i.e. a rule of substitution for the variables **E**, **X** and **Y** on whose application the enumerated forms would become sentences in everyday language, and a principle of modus ponens or detachment (a rule to the effect that if two sentences of the form " If **X**, then **Y** " and " **X** " are true, then a sentence of the form " **Y** " is true).

Some logicians seem to have confused the rules of Chwistek's formal system with those of his "intuitive" logic.[1] They maintain that Chwistek's failure to distinguish between an object language and its syntax language leads him to confuse the concepts of these two languages. However, a careful examination of Chwistek's system reveals that while Chwistek includes an object language and its syntax language within the language of semantics, the concepts of these two sub-languages of semantics are by no means used interchangeably.

Nevertheless it seems that Chwistek gains no real advantage by relying upon this "intuitive" logic. His reason for proposing that such a logic be made the basis of all further study is two-fold. In the first place it is the simplest possible "logic" with which he is familiar.[2] Secondly it is a device which serves to eliminate the need for an infinite hierarchy of languages required when a distinction is drawn between a language and its "metalanguage".[3] However, it would seem that the relatively simple form of "intuitive" logic complicates the structure of the language of semantics, Chwistek's main concern. On the other hand, with the help of the distinction between a language and its "metalanguage", it is certainly possible to obtain precision, to eliminate in actual practice all reliance upon the infinite hierarchy of languages to which Chwistek objects, and to obtain relatively simpler systems than Chwistek's system of semantics. Moreover in a certain sense Chwistek's "intuitive" logic functions as a "metalanguage".

Chwistek calls his formal system, which is governed not only by the rules of "intuitive" logic but by the principles of sound reason, the "fundamental system of semantics". This system contains three systems, which are specified by laying down appropriate rules for constructing expressions and for deriving theorems. The set of rules (rules for

[1] Cf. Professor W. V. Quine's review of *N.F.F.M.* in *The Journal of Symbolic Logic*, vol. 3, 1938, pp. 120–1. Dr. Quine fails to note that Chwistek offers two distinct rules of substitution, one rule, which is part of his formal system, which governs the substitution of constant expressions for constant expressions in constant expressions, and a second and more familiar rule formulated in his "intuitive" logic, concerning the substitution of constants for variables in functions.

[2] "It" ['intuitive' logic] "is really the poorest logic I know," says Chwistek in a letter dated 20th July, 1939.

[3] i e. the distinction between a language and its "metalanguage" requires a similar distinction between the "metalanguage" and its "meta-metalanguage", etc.

expressions), denoted by $(R\ E)$, is sufficient to construct all the expressions with which semantics deals. The set of rules (rules for the auxiliary system), denoted by $(R\ A)$, is used in deriving the theorems of the calculus of propositions. The set of rules (rules for proper systems), denoted by $(R\ P)$, is utilized in deriving theorems of the utmost importance in the calculus of classes and in certain portions of mathematics.

THE SYSTEM OF EXPRESSIONS (E)

The system of expressions (E) is obtained by applying the rules for expressions $(R\ E)$ and the rules of "intuitive" logic. When this system is regarded merely as a set of theorems obtained by applying these rules, Chwistek's construction presents no real difficulties. The rules for expressions $(R\ E)$ consist of two rules which permit the construction of an unlimited number of expressions, and are formulated as follows:

"**c** *is an expression,*"

"If **E** *and* **F** *are expressions,* $*$**E F** *is an expression.*" The following expressions are typical of those which can be derived by the application of these rules: "**c**," "$*$**c c**," "$**$**c c**$*$**c c**," "$*$**c**$*$**c c**," "$**$**c c c**." While these rules govern the formation of all expressions, they do *not* constitute a general definition of expressions since the first rule is itself a statement of semantics.[1]

Obviously but four signs are involved in the rules $(R\ E)$, "$*$," "**c**," "**E**," and "**F**". The sign "**c**" must be a constant expression, for if on the basis of the first rule a sign such as "§" were substituted for "**c**", an expression would not be obtained. On the other hand the letters "**E**" and "**F**" must be regarded as variables.[2] Otherwise the second rule

[1] In this connection it is somewhat difficult to accept Chwistek's conventional decision to interpret signs as expressions (85), for although "**c**" is an expression, "$*$" by itself is not an expression although it is a sign. Chwistek uses the term "segment" to denote any series of consecutive signs which are constituents of an expression. The fact that a segment of an expression is not necessarily an expression, but may contain any number of signs (88), makes it impossible to prove a number of familiar syntactical theorems concerning the expressions of the fundamental system of semantics.

[2] It is curious that in his fundamental system of semantics, Chwistek insists that variables which denote starred expressions are to be printed in the same kind of type as are the constant expressions themselves In his earlier chapters Chwistek imposes no such restriction The consequence of this procedure is that the distinction between a variable and a constant expression is not immediately obvious.

in combination with the first would not permit the construction of an unlimited number of expressions. However, the domain of substitution of these variables is restricted to expressions and the variables themselves do not appear in the theorems of semantics. The elimination of these variables is possible (166) only by the use of verbal definitions. However, these definitions also contain some signs and their use would involve a series of detailed descriptions.[1] Thus it may be concluded that the second rule supplies the pattern [2] " $*$**EF** " [3] of all expressions with the exception of the expression " **c** ".

In general the theorems of the system of semantics can easily be derived by the application of carefully specified rules. Nevertheless considerable diversity of opinion arises in connection with the proper interpretation of these theorems. Differences of opinion can be found even at this early stage [4] in connection with the interpretation of the two-place star operator. Chwistek himself offers two distinct interpretations of this sign. In one context he maintains that it is used as a parenthesis,[5] i.e. as a punctuation mark. On this interpretation semantics might well be regarded as an object language constructed by the iteration of a single sign " **c** ". It could not, however, contain any syntactical theorems. In another context Chwistek interprets the star operator as the sign of juxtaposition of expressions (84). From this point of view semantics could be interpreted as a syntax language based on two primitives, " $*$ " and " **c** ". Unfortunately for this interpretation some of the theorems actually derived by

[1] Cf. *K.P.Z.*, pp. 292–3.

[2] In algebra it is customary to represent the general form of a linear equation as follows: $Ax + By + C = 0$. The letters " A ", " B ", and " C " denote any constants, the letters " x " and " y " any variables, " $+$ " and " $=$ " are mathematical operators, and " 0 " denotes the number 0. From this general form or *pattern* it is possible to derive any number of linear equations by assigning values to the letters " A ", " B " and " C ". Analogously a pattern is a device used in semantics to indicate the form of expressions. The pattern " $*$**EF** ", for example, is used to indicate the form of all the expressions of semantics. The sign " $*$ " is a semantical operator, the letters " **E** " and " **F** " are variables, for which it is possible to substitute any expression. From this pattern it is possible to obtain an unlimited number of expressions.

[3] Although there is considerable diversity of opinion concerning the interpretation of " $*$ ", it is possible to construe this sign as an operator. It should be observed that Chwistek places all operators before rather than between the expressions which they govern. This is done in order to avoid the use of such punctuation marks as brackets and parentheses.

[4] i.e. in the system of expressions (E).

[5] Cf. *N.F.F.M.*, p. 2.

Chwistek turn out to be theorems of an object language. It would therefore seem that semantics is an object language based upon two primitives, " $*$ " and " c ". Unfortunately, however, some of the theorems of semantics are syntactical theorems. Although the possibility that semantics is a new device which states the syntax of a language within that language still remains to be eliminated (297–8, 310), it should be said that semantics must be interpreted as a language based upon two primitives, " $*$ " and " c ", which consists of syntactical statements and statements in its object language. On this interpretation the star operator would be used as the sign of juxtaposition in the syntactical statements and either as an operator [1] or as a punctuation mark in the theorems of the object language.

THE AUXILIARY SYSTEM (*A*)

It has been pointed out that it is possible to construct an unlimited number of expressions by the application of the rules for expressions (*R E*). However, while "c" is the fundamental element of all expressions, " c " itself has no meaning.[2] Similarly expressions which are combinations of the signs " c " and " $*$ " have no meaning. However, when Chwistek develops the auxiliary system (*A*), i.e. the system obtained by applying the rules for the auxiliary system (*R A*) and the rules of " intuitive " logic, he assigns meaning to certain expressions of the latter kind in order to develop the elementary logical calculi.[3]

The actual formulation of the rules for the auxiliary system (*A*) is given in terms of familiar [4] syntactical, logical and mathematical concepts, which are correlated with certain of the expressions of the system of expressions (*E*). This correlation is presented in a table of abbreviations (166–7). Abbreviations function in Chwistek's system as nominal definitions and consequently can in theory be eliminated. Since, however, only expressions which have abbreviations are regarded as meaningful, the interpretation of semantics as a device which states the syntax of a language

[1] This operator has never been interpreted.

[2] Cf. *N.F.F.M.*, p. 2.

[3] The set of rules (*R A*) is also sufficient to develop what Chwistek calls the elementary semantical calculus.

[4] An occasional new concept is introduced, e.g. the concept of substitution (300).

within that language must be rejected. It should also be pointed out in connection with the table of abbreviations, that reference to a specific system is always implied although not specifically stated.[1] For example, " **.0 L** " is an abbreviation of " * **L L** " only in the auxiliary system (*A*). Moreover since the rules for the auxiliary system govern only those concepts of the system of expressions (*E*) which are considered in the table of abbreviations, the auxiliary system (*A*) deals with a sub-class of the expressions of the system of expressions (*E*).

It is in connection with the auxiliary system (*A*) that Chwistek first introduces the theory of types into the system of semantics. It should be recalled that two distinct theories of types, the simple and the branched theory, were proposed for the elimination of the paradoxes of the theory of classes (xxxiv–xxxv). Chwistek himself at different times advocated each of these theories (xxxv–xxxviii). Nevertheless his awareness of the difficulties involved in both these theories led him to seek a new theory which would eliminate all the paradoxes and yet avoid these difficulties.[2] The concepts involved in this new theory, which may be called the semantical theory of types, will now be explained.

A type is ascribed to every expression of the auxiliary system.[3] Types are indicated by the expression " **c** " and the natural numbers. The general rules for determining the type of an expression are the following: Any expression containing a star and two expressions of type **K** is an expression of type **K**. For example, " * **c c** " is an expression of type **c**. Any expression **.0 K** is an expression of type **K**. Consequently the expression " * * **c c** * **c c** " is an expression of type **0**, since **0** is defined as * **c c**. Any expression of type **K**, which is not an expression of type **.0 K**, is an original expression of type **K**.[4] This does not mean that an original expression of type **K** cannot also be an expression of type **K**. Thus

[1] Cf. *K.P.Z.*, p. 295.

[2] For example, the assumption of the existence of individuals and of an infinite number thereof (152–5), which characterizes the attempt to eliminate the paradoxes by means of Russell's simple theory of types, is unnecessary in view of the fact that semantics deals solely with constructible expressions. The system of semantics contains no " ideal " objects On the other hand it is possible to formulate the axiom of reducibility of the branched theory of types (156) in modified form (189–190) in consequence of the particular concept of type which Chwistek utilizes (170).

[3] However, in Chwistek's elementary considerations reference to type is implicit. The only expressions considered are of type **c**.

[4] " Original expression " is a technical term defined as indicated.

"**0**" is both an expression of type **c** and an original expression of type **c**. It is an expression of type **c** because it is composed of a star and two expressions of type **c**. But it is also an original expression of type **c** since it is not an expression of type **0**. The type of an expression is fixed in a given context, although from the pattern of an expression, all of whose constituents are variables, it is possible to derive further *patterns*, each of which has a different type. The type of a pattern then is independent of the type of the variables contained therein. The most interesting feature of Chwistek's theory of types is the fact that an expression of type **.0 K** is simultaneously an expression of type **K**. For example, the expression "**1**", i.e. "****cc*cc**", is an expression of type **0** and an expression of type **c**.[1] It should be noted in this connection that just as the theory presented is a theory of *semantical* types, the paradoxes in question become semantical paradoxes. The rules of this theory of types are not, however, contained among the rules for the auxiliary system (*R A*) (168).

The following comments upon some of the important concepts of the auxiliary system (*A*), introduced in the table of abbreviations (166–7), is designed to be of aid in understanding the development of the fundamental system of semantics. Each of these concepts is introduced as a pattern for expressions of the auxiliary system (*A*). These patterns contain the variables: **E, F**, . . . which, although they themselves do not belong to the system of semantics, denote constants of this system. The patterns considered are of two types: **L** and **c**. However, the patterns of type **L** are not constituents of the rules for the auxiliary system (*R A*). Their only function is to permit the derivation of patterns of type **c**. The actual expressions of the auxiliary system (*A*) are all of the latter type.

The fundamental concept of the auxiliary system (*A*) is the concept of integers.[2] Integers are denoted by the patterns:

[1] It is this characteristic which permits the formulation of the axiom of reducibility.

[2] It is this fact that suggests the possibility that the technique of semantics has some analogies to the process of arithmetizing a language. Just as it is possible to arithmetize geometry by introducing numerical co-ordinates in such a way that every geometrical configuration can be replaced by relations between numbers, it is possible to correlate with every expression of a language a number in such a way that every statement of the language can be replaced by an arithmetical relation. The first extensive use of the process of arithmetizing a language was made by Gödel.

.0 L, **.1 L**, . . ., where "**L**" denotes the type of the integer. It develops that any integer of type **L** is not only reducible in practice to an integer of type **c**, which Chwistek calls a fundamental integer, but frequently must be so reduced if any sense is to be assigned to certain of Chwistek's procedures. Yet in Chwistek's system the latter concept is defined in terms of the former. This is indeed a curious state of affairs. Chwistek introduces certain auxiliary expressions which he calls asymmetrical expressions. These expressions are denoted by the patterns: **.I L** and **.II L**.[1] With their help he is able to reduce the calculus of propositions to the system of semantics. This procedure stands in marked contrast to Chwistek's elementary considerations where the semantical and logical expressions are sharply distinguished.

Chwistek now defines the fundamental concepts of the propositional and semantical calculi, the pattern of the stroke operator and the pattern of substitution. It should be noted that neither of these concepts are primitive concepts in Chwistek's system. The stroke operator, originally introduced by Sheffer, is represented by the pattern: **/ E F**, which is read "Not both **E** and **F**". With the aid of this pattern the entire calculus of propositions can be developed. It should be noted that Chwistek permits the substitution of semantical as well as logical expressions for the variables of this pattern. He also introduces a new pattern which he calls the pattern of substitution. It is denoted by the symbol: **(E F G H)**, which is read

"**H** is the result of the substitution of **G** for **F** in **E**".

This pattern, which concerns the substitution of constant expressions for constant expressions in constant expressions, is the basic concept of the semantical calculus and is defined solely in terms of semantical concepts.

Chwistek's introduction of the pattern of semantical identity: **= E F**, which is to be interpreted as "**E** is identical with **F**", is of special interest because this concept has an important function in the development of arithmetic. The fundamental role of integers in the system of semantics has already been noted. One consequence of this fact is that Chwistek does not need to draw a distinction between the identity of integers and semantical identity.

Chwistek also makes use of the concept of inclusion.

[1] Reference must be made to the table of abbreviations (166–7) for exact definitions of the various concepts under discussion.

This concept is denoted by the pattern "{ **E F** }" and may be read "**F** is contained in **E**". For example, "**c** is contained in ∗ **c c**" is denoted by the symbol : { ∗ **c c c** } or the symbol : { **0 c** }. It is obvious that "∗ **c c**" is contained in "∗ **c c**". However, since Chwistek defines the concept of inclusion in terms of the pattern of substitution, and the patterns of logical negation and logical alternation, it is not a purely semantical concept. Some logicians, among them Chwistek's pupil Hetper, maintain that the goal of semantics is the derivation of all possible theorems from the concept of inclusion and the concepts of the logical calculus by the application of the rules for the auxiliary system (*R A*).[1] On this interpretation inclusion is a purely semantical operation. However, this interpretation makes Chwistek's reduction of logical concepts to semantical ones pointless, since there can be but one purely semantical concept. Moreover no use could be made of such other semantical expressions as the auxiliary asymmetrical expressions. Furthermore Chwistek's reduction of logical concepts to semantical ones is not made merely for the sake of elegance. It has a definite positive function which is exemplified in the elimination of the ambiguities involved in the concept "proposition".

It is of some interest to observe that the word "expression" never occurs either in the auxiliary system (*A*) or in the proper systems (*P*), which are derived by applying the rules for proper systems (*R P*) and the rules of "intuitive" logic. It can therefore be inferred that the rules for expressions (*R E*) are used primarily as a means of identifying expressions. The concept : **Expr E** of the auxiliary system (*A*), which is defined by the pattern "= **E E**" and which may loosely be rendered as "**E** is an expression", is introduced for a different purpose. It is used in the derivation of theorems of the auxiliary system (*A*), e.g. in the derivation of the "axioms" of semantical identity. Similarly "**Prop E**", which is defined by the pattern: ⊃ **E E**, and which may be read

"**E** is a proposition",

is used in the derivation of the "axioms" of logical identity.

[1] Although the concept of inclusion is not a purely semantical concept in the system presented in the text, this fact is unimportant because Hetper has shown that the pattern of substitution can be derived from that of inclusion and these same logical concepts. Cf. W. Hetper : "Podstawy semantyki" ("Foundations of Semantics"), *Wiadomości matematyczny*, t. xliii, 1936, pp. 82–6.

As has already been pointed out, the rules for the auxiliary system (*A*) are sufficient for the development of the elementary calculus of propositions. While in the main Chwistek wishes to retain the usual results of this calculus, in his construction of the auxiliary system (*A*), the concept "function" disappears, not because there is no essential difference between the operations performed on functions and those performed on propositions, but because the auxiliary system (*A*) is constructed in such a way that only expressions of type **c** appear. Variables are found in the rules of procedure (*R A*), i.e. in the patterns involved therein, but never in the propositions derived by the application of these rules. Consequently in the auxiliary system (*A*), such a fundamental principle as the axiom of logical identity takes the form of a series of theorems in which all the constituent expressions are of type **c** and in which the desired property is attributed to certain of the constituent expressions. Thus it can be shown that:

⊃ 0 0 *is a theorem,*
⊃ 1 1 *is a theorem,*
⊃ 2 2 *is a theorem,*
⊃ (0 0 0 0) (0 0 0 0) *is a theorem,* etc.,

To demonstrate theorems of this kind Chwistek finds it necessary to introduce the new concept of substitution, represented by the pattern **(E F G H)** (300). This kind of substitution does not replace the substitution of constants for variables (89–90, 167) but is a supplementary device. However, in the auxiliary system (*A*) Chwistek finds it impossible to prove that:

⊃ p p *is a theorem.*

Chwistek develops the calculus of propositions with the help of the more basic science, semantics, because of the disagreement prevailing among philosophers and logicians concerning the nature of propositions (110–1). He wishes to retain the defining characteristics proposed by Aristotle for propositions, i.e. the possession of one of the properties, truth or falsity,[1] but finds this possible only if propositions have completely and precisely determined meanings. Since he maintains that it is possible to obtain such propositions only if they are defined in terms of constructible expressions,

[1] Chwistek takes this position in view of his insistence upon the principle of contradiction as one of the basic principles of sound reason.

Chwistek suggests that it is necessary to develop the theory of propositions in terms of the concepts of semantics.

In a letter dated May 28, 1939, Chwistek writes as follows concerning the interpretation of propositions (169): " Such symbols as **(0 0 0 0)** have no individual meaning. They are not names at all. To have significant propositions we must assume that **(0 0 0 0)** is true " [or false] " or that it is a theorem ". Since, however, the concepts " truth " and " falsity " are absent in the final interpretation of a proposition, they are auxiliary ideas employed only in the actual process of interpreting a proposition. It is therefore clear that Chwistek supplements the language of semantics, by a special language, which he calls the " language of interpretation ". Similar remarks are, of course, relevant to all the patterns and expressions of the auxiliary system (A).

In the " language of interpretation " the concepts " true symbolic proposition " and " false symbolic proposition " are regarded as primitive.[1] The application of the rules of this language to the auxiliary system (A) seems to take place in two steps. The first step involves the interpretation of

" **Prop (0 0 0 0) is a theorem** " as " **(0 0 0 0) is a true symbolic proposition** ".

The second step is the interpretation of the latter result as

" **0 is the result of the substitution of 0 for 0 in 0** ".

It should be noted, however, that the proposition

" **Prop (0 0 0 0)** *is a theorem* "

must be distinguished from the proposition

" **(0 0 0 0)** *is a theorem* "

even though both these propositions have the same interpretation.[2] Similar complications arise in connection with the interpretation of " **Expr 0** *is a theorem* ".

In spite of this difficulty Chwistek uses the rules of the " language of interpretation " to support his contention that it is unnecessary to distinguish an object language and its syntax language by means of a specially introduced symbolism. His position is developed with the help of an illustration borrowed from everyday language, which employs

[1] Cf. letter of August 25, 1939.

[2] Cf. *N.F.F.M.*, p. 5. Changes in type have been made in conformity with the typographical rules indicated in the preface.

both French and English.[1] The reader is asked to consider the proposition:

> "Socrate est un homme" est une proposition vraie, puisque Socrate est un homme.

If this proposition is translated into English it becomes:

> "Socrate est un homme" is a true proposition because Socrates is a man.

Chwistek points out that in the translated proposition the sentence "Socrate est un homme" appears only once although in the original proposition it appears twice. He suggests that analogously when

"**/ ((0 0 0 0) 0 0 0) (0 0 0 0)** *is a theorem*"

is properly interpreted,[2] the expression "**(0 0 0 0)**" likewise appears only once. He maintains that since no confusion is possible under these circumstances, there is no need to use quotation marks to make the meaning of this proposition clear. However, even if it is granted that no confusion is possible concerning the interpretation of the proposition under consideration, another proposition can easily be constructed, the meaning of which is not immediately obvious without the help of a distinction between a language and its "metalanguage". For it is impossible to give an adequate interpretation of the proposition:

"**/ Prop (0 0 0 0) (0 0 0 0)** *is a theorem*"

on the basis of the rules presented by Chwistek. Even a cursory glance at this proposition reveals that its two constituents have the same interpretation. Thus Chwistek's claim that no confusion can possibly arise in connection with the interpretation of the propositions of semantics is invalid.

With the help of the apparatus already set up Chwistek is in a position to define the rational numbers and to construct the patterns for operations upon both integers and rational numbers (94–9). But he is not able to prove the theorems

[1] Cf. *F.S.E.*, pp. 90–1.

[2] "**It is not the case that 0 is the result of substituting 0 for 0 in (0 0 0 0) and that 0 is the result of substituting 0 for 0 in 0**" is the interpretation of "**/ ((0 0 0 0) 0 0 0) (0 0 0 0)** *is a theorem*".

he discusses in this connection, because the rules for the auxiliary system ($R\,A$) include no rule of (semantical) induction.[1] Moreover since Chwistek's definition of classes depends upon the use of quantifiers, he cannot as yet construct patterns for real numbers. However, there seems to be no good reason why quantifiers, propositions containing quantifiers and classes could not be discussed at this stage. Of course the discussions of these subjects which could be conducted at this time would not be the most general ones possible, since only specific constant expressions could be considered. For example, while Chwistek could derive the theorem: $\mathbf{= + 1\,1\,2}$, he could not derive the theorem: $= + \,a\,a\,\mathbf{2}\,a$, where "$a$" denotes any integer.[2]

THE PROPER SYSTEMS (P)

With the introduction of the proper systems (P),[3] Chwistek passes to more interesting and original lines of thought. These proper systems are developed in such a way that while their expressions belong both to the system of expressions (E) and to the auxiliary system (A), a distinctive symbolism is employed which indicates that in this context they are to be regarded as expressions of the proper systems (P). The abbreviations and rules are formulated in terms of the real variables "$\mathbf{M}$", "$\mathbf{N}$", . . ., which denote expressions, although the proper systems themselves do not contain these variables. It is impossible to give a complete analysis of proper systems within the limits of this appendix. The few hints given concerning their interpretation and content are designed merely to aid the reader in understanding this difficult portion of the text. The discussion will be conducted in terms of elementary systems, which are sub-systems of proper systems. An explanation will be given only of *patterns* of proper systems, although illustrative material will be drawn from specific proper systems.

Where "$\mathbf{M}$" and "$\mathbf{N}$" are integers, and "$\mathbf{L}$" is either

[1] i.e. Chwistek cannot show that if any number has a given property, its successor also has this property.

[2] In ordinary mathematical notation, these theorems would be written: $1 + 1 = 2$ and $a + a = 2\,a$ respectively.

[3] The proper systems (P) are systems obtained with the help of the rules for proper systems ($R\,P$) and the rules of "intuitive" logic.

the expression "$\mathbf{c}$" or an integer, if $\mathbf{N}+1$ is contained in $\mathbf{M}$ and if the integer of the form: $*\mathbf{L}\,\mathbf{L}$ is contained in $\mathbf{N}$, then "$\models [\mathbf{M}\,\mathbf{N}]\,\mathbf{L}$" is an elementary system.[1] Consequently it is possible to construct elementary systems whenever $\mathbf{M} \geq \mathbf{N}+1$, and $\mathbf{N} \geq *\mathbf{L}\,\mathbf{L}$. For example, "$\models [\mathbf{1}\,\mathbf{0}]\,\mathbf{c}$" is an elementary system since $\mathbf{1} = \mathbf{0} + 1$, and $\mathbf{0} = *\,\mathbf{c}\,\mathbf{c}$. It is obvious that a proper system: $[\mathbf{M}\,\mathbf{N}]$ contains all elementary systems which have the same values for the variables "$\mathbf{M}$" and "$\mathbf{N}$". Thus the proper system: $[\mathbf{2}\,\mathbf{1}]$ contains the elementary systems: $\models [\mathbf{2}\,\mathbf{1}]\,\mathbf{c}$ and $\models [\mathbf{2}\,\mathbf{1}]\,\mathbf{0}$.

Variables are now introduced into the system of semantics (171).[2] These variables are of two kinds, semantical and logical. On the basis of the table of abbreviations the distinction between them is obvious, although in practice it is not clear (111–12). Every semantical and logical variable is either a real variable or an apparent variable. The symbols or patterns: $\alpha_{\mathbf{K}\,\mathbf{L}}, \beta_{\mathbf{K}\,\mathbf{L}}, \ldots$ and $\mathbf{a}_{\mathbf{K}\,\mathbf{L}}, \mathbf{b}_{\mathbf{K}\,\mathbf{L}}, \ldots$, which are abbreviations of the patterns "$**.\mathbf{I}\,\mathbf{K}\,\mathbf{1}_{\mathbf{K}}.\mathbf{I}\,\mathbf{L}$", "$**.\mathbf{I}\mathbf{K}\mathbf{2}_{\mathbf{K}}.\mathbf{I}\mathbf{L}$", ... and "$**.\mathbf{II}\mathbf{K}\mathbf{1}_{\mathbf{K}}.\mathbf{I}\mathbf{L}$", "$**.\mathbf{II}\mathbf{K}\mathbf{2}_{\mathbf{K}}.\mathbf{I}\mathbf{L}$", ... are said to denote variables of the system: $\models [\mathbf{M}\,\mathbf{N}]\,\mathbf{L}$ provided that $*\mathbf{L}\,\mathbf{L} \leq \mathbf{N}+1 \leq \mathbf{K} \leq \mathbf{M}$.[3] "$\mathbf{K}$" is said to indicate the type of the variable. But these variables are also original expressions of type $\mathbf{L}$ (175). Thus "$\mathbf{x}_{1\,\mathbf{c}}$" is a variable of the system: $[\mathbf{1}\,\mathbf{0}]$, since $*\,\mathbf{c}\,\mathbf{c} < \mathbf{0} + 1 = \mathbf{1} = \mathbf{1}$. The type of this variable is $\mathbf{1}$.[4] On the other hand this variable is also an original expression of type $\mathbf{c}$.

On the subject of variables Chwistek comments as follows: "The use of expressions as variables is connected with types. For example, $\mathbf{x}_{1\,\mathbf{c}}$ is a variable in the symbolic proposition $= \mathbf{x}_{1\,\mathbf{c}}\,\mathbf{x}_{1\,\mathbf{c}}\,[\mathbf{0}]$, but it is a constant expression in the symbolic proposition $= \mathbf{x}_{1\,\mathbf{c}}\,\mathbf{x}_{1\,\mathbf{c}}\,[\mathbf{c}]$" "because it cannot be discussed

[1] The purpose of this account is to enable the reader to understand Chwistek's general intent with respect to proper systems. It is not a rigorous account, but rather an interpretation or translation of Chwistek's views. Consequently free use is made of familiar terminology, especially of arithmetical notions. Such notions are readily identifiable since ordinary mathematical conventions concerning the type and order of symbols are followed.

[2] These variables must not, of course, be confused with the variables "$\mathbf{M}$", "$\mathbf{N}$", ... which are not variables of the proper systems (P) itself, although they are found in the rules governing these systems, and in the patterns of the variables now under consideration.

[3] Symbols which satisfy this condition are called "real variables", provided they do not also satisfy the condition indicated below (308).

[4] Obviously then the expressions of the proper systems (P) need not be of type $\mathbf{c}$ as was the case in the auxiliary system (A).

by means of axioms of type **0** ".[1] In other words the symbol "$\mathbf{x}_{1\,\mathbf{c}}$" is not consistently used to denote a semantical variable. Whether it is a variable or constant seems to depend upon whether **c** is less than or equal to the type of the expression of which it is a constituent. To decide whether this symbol represents a variable or a constant expression, it is therefore necessary to examine the context in which it occurs. If it is not a variable but a constant, the peculiar function of this original expression of type **c** is not obvious. If it is a variable it must also be regarded as an expression of type **c**. In other words the same symbol is used to represent an expression and what is denoted by that expression. Furthermore the sense in which the component of a variable, e.g. " **.I L** " or " **.II L** ", cannot be an expression of a proper system (176) is not clear. For even when such a symbol as "$\mathbf{a}_{1\,\mathbf{c}}$" is interpreted as a variable, it is an abbreviation of an expression containing the expressions " **.II 1** " and " **.I c** " as constituents.[2]

It is now possible to enumerate the patterns of expressions of elementary systems. If " **⊨ [M N] L** " is an elementary system, " **.1 N** " is a pattern of expressions of this system. If " **E** " is a logical or semantical real variable of this system, it is a pattern of expressions of this system. Finally if " **E** " and " **F** " are patterns of expressions of this system, so is " ***EF** ". The theorems of the elementary system: **⊨ [M N] L** and therefore of the proper system: **[M N]** are represented by the pattern " **⊨ [M N] E** ", where 'E" stands for any of the various patterns which have been considered. Consequently " **.10** ", "$\mathbf{a}_{1\,\mathbf{c}}$", "$\alpha_{1\,\mathbf{c}}$", " ***.10** $\mathbf{a}_{1\,\mathbf{c}}$" and " ***.10 .10** " are all expressions of the system " **⊨ [10] c** ", and " **⊨ [10] .10** *is a theorem* " is a theorem of this system. It is apparently possible to substitute for the variables any integer, any constant expression denoted by "$\mathbf{a}_{\mathbf{KL}}$", "$\mathbf{b}_{\mathbf{KL}}$",... or "$\alpha_{\mathbf{KL}}$", "$\beta_{\mathbf{KL}}$", ..., or any combination of these constants which contains the star operator, e.g. any integers greater than " **.1 N** ".[3]

[1] These passages are taken from Chwistek's letter of May 28, 1939, and from a letter undated, but written some time between March 4 and March 28, 1939.

[2] In Chwistek's earlier writings the symbols: **.I L** and **.II L** do not occur. The symbols: *I* (*L*) and *II* (*L*) are used in their place. Similarly " .0 (*L*) " is used instead of " **.0 L** ". No distinction is drawn between **.0 L** and $\mathbf{0}_{\mathbf{L}}$ Thus a change not only with regard to typography but with regard to symbolism is introduced in the present work.

[3] Cf. *F.P.G.T.*, p. 68.

Presumably these expressions would be the values of the variables, since theorems containing the symbol: $\mathbf{a}_{1\,c}$ are interpreted as theorems concerning the values of a variable (175–6) in certain cases. In such cases the following condition is imposed: the type of this symbol must be the same as the type of the entire expression or proposition of which it is a constituent. The word "presumably" is used advisedly since Chwistek has never specifically defined the concept: value of a variable.

In Chwistek's elementary considerations, barred symbols denote apparent variables. They are distinguished from the real variables by the imposition of the further condition that "$.0\mathbf{L}$" replaces "$\mathbf{K}$". Consequently $\mathbf{L}+1=\mathbf{N}+1=\mathbf{K}$, i.e. $\mathbf{L}=\mathbf{N}=\mathbf{K}-1$.[1] For example, the system: $\models [\,2\,0\,]\,c$ contains the propositions with the apparent semantical variables: $\bar{\mathbf{a}}_0$, $\bar{\mathbf{b}}_0$, . . ., i.e. $\mathbf{a}_{1\,0}$, $\mathbf{b}_{1\,0}$, . . . as constituents. It is clear that in semantics the same symbol does not function both as a real variable *and* as an apparent variable.[2] For example, in the system: $[\,1\,0\,]$, the real variables are denoted by the symbols: $\mathbf{a}_{1\,c}$, $\mathbf{b}_{1\,c}$, . . ., $\alpha_{1\,c}$, $\beta_{1\,c}$, . . ., while the apparent variables are denoted by the symbols: $\mathbf{a}_{1\,0}$, $\mathbf{b}_{1\,0}$, . . ., $\alpha_{1\,0}$, $\beta_{1\,0}$, ...[3]

According to Chwistek while "$\Pi\,[\,2\,1\,]\,\alpha_{2\,1}\alpha_{2\,1}$" is not an expression of the system: $[\,2\,1\,]$, it is an expression of the system: $[\,2\,0\,]$. Consequently although "it cannot be discussed in $[\,2\,1\,]$, it can be the object of metamathematical research in $[\,2\,0\,]$" (177). However, the expression: $\alpha_{2\,1}$ is an apparent variable of both "$[\,2\,0\,]$" and "$[\,2\,1\,]$". The expression "$\Pi\,[\,2\,1\,]\,\alpha_{2\,1}\alpha_{2\,1}$" can therefore be reduced to a combination of the patterns of expressions of either system. The situation is different in the case of the proposition:

$$\exists\,[\,1\,0\,]\,\alpha_{1\,0}\wedge(=\mathbf{a}_{2\,1}\,\mathbf{a}_{2\,1}\,[\,2\,]\,\mathbf{a}_{2\,1}\,.0\,1\,\alpha_{1\,0})[\,1\,]\,\alpha_{1\,0}$$ [4]

[1] The apparent variables: $\mathbf{a}_{K\,L}$, $\mathbf{b}_{K\,L}$, . . ., $\alpha_{K\,L}$, $\beta_{K\,L}$, . . . of the system: $[\,\mathbf{M}\,\mathbf{N}\,]$ therefore satisfy the following two conditions:

(1) $*\mathbf{L}\,\mathbf{L}\leq\mathbf{N}+1\leq\mathbf{K}\leq\mathbf{M}$

(2) $\mathbf{L}=\mathbf{N}=\mathbf{K}-1$

[2] In a single system

[3] This notation should be contrasted with that of other logicians who employ the same symbol to denote both real and apparent variables. The latter usage of a variable is distinguished by the presence of a quantifier. For example in the function $f(x)$, "x" is a real variable, while in the function: $\Pi x f(x)$, where "Πx" is the universal quantifier "for all x," "x" is an apparent variable. In Chwistek's system the symbol denoting a real variable differs from the symbol denoting an apparent variable. Furthermore, each proper system makes use of certain variables and no others.

[4] *N.F.F.M.*, p. 13.

of system " **[20]** ", for this system does not contain the real variable: $\mathfrak{a}_{2\,1}$. Consequently this proposition cannot be reduced to any of the patterns of expressions of this system. Thus on the basis of the second illustration it does seem possible to avoid paradoxes, although if the expression: $\mathfrak{a}_{2\,1}$ were employed as an apparent variable, the resultant proposition could be reduced to the patterns of expressions of system: **[20]**. It therefore follows that Chwistek's contention: " an infinite sequence of systems of metamathematics[1] can . . . be constructed such that each subsequent system can be investigated in the preceding system " (164), is irrelevant. Metamathematical research (in any sense whatsoever) is not involved here, because it would be impossible to apply the rule of systems [rule 3.4] (174), since there is no proper system: **[2c]**.

The notion of a propositional function differs somewhat from the usual meaning of this term. From a propositional function it is possible to derive by substitution either another propositional function or a proposition. On the other hand no expression containing the symbol: **.I L** is a proposition of the system: $\models$ **[M N] L**. Moreover on the basis of the restrictions Chwistek imposes upon variables, many expressions which at first sight appear to be propositions actually are not.[2] It would also seem that " semantical " as well as logical propositions are possible.[3]

Chwistek has now introduced the apparatus necessary for the development of the calculus of propositions and the calculus of quantifiers in the proper systems (*P*). It is hardly necessary to mention that each proper system has its own set of calculi. As a matter of fact each proper system

[1] The meaning of this concept will be considered below (315 ff).

[2] For example " $\models$ **[10] Π [1c]** $\mathfrak{a}_{1\,c}\,\mathfrak{a}_{1\,c}$ " is not a proposition, because " $\mathfrak{a}_{1\,c}$ " is not an *apparent* variable of the system: **[10]**.

" $\models$ **[20] Π [10]** $\bar{\mathfrak{a}}_{1\,0} = \bar{\mathfrak{a}}_{1\,0}\,\bar{\mathfrak{a}}_{1\,0}$ **[0]** "

is not a proposition, because " $\bar{\mathfrak{a}}_{1\,0}$ " is not a variable but a constant and consequently " $= \bar{\mathfrak{a}}_{1\,0}\,\bar{\mathfrak{a}}_{1\,0}$ **[0]** " is not a propositional function On the other hand the quantified propositional function:

$\models$ **[20] Π [10]** $\bar{\mathfrak{a}}_{1\,0} = \bar{\mathfrak{a}}_{1\,0}\,\bar{\mathfrak{a}}_{1\,0}$ **[1]**,

is a proposition.

[3] It is possible to substitute for the real variables of the system: **[10]** any integer greater than **2**, or any expression of the form " *** E F** ", where " **E** " and " **F** " are integers greater than **2**. Consequently in

" $\models$ **[10]** $\mathfrak{a}_{1\,c}$ *is a theorem* "

it would be possible to substitute for " $\mathfrak{a}_{1\,c}$ " an expression of the form: *** * * * .I c .0 E .0 F .0 G .0 H**, where " **E** ", " **F** ", " **G** ", " **H** " are integers. In that case one would obtain a semantical proposition of the form:

$\models$ **[10] (E F G H) [c]** *is a theorem.*

contains two distinct calculi of propositions. The first of these calculi is worked out for the particular constant expressions of the system. It is possible, for example, to derive : $\models$ **[10]⊃33** *is a theorem.* There is no theorem of this kind in the ordinary calculus of propositions, although Chwistek develops analogous theorems in the auxiliary system (*A*). The second calculus of propositions is more in conformity with what is usually meant by this term, since it contains variables as well as constants. For example, it is possible to derive : $\models$ **[10]⊃** $\mathfrak{a}_{1\,0}\,\mathfrak{a}_{1\,0}$ *is a theorem,* a theorem which is analogous to the theorem : ⊃*p p* of the usual calculus of propositions. In this calculus if " $\mathfrak{a}_{1\,0}$ " and " $\mathfrak{b}_{1\,0}$ " are constituents of the same expression, they are two distinct variables. This fact is indicated in the table of abbreviations (171) by the correlation of a definite expression with each variable. Unfortunately, however, this correlation is entirely arbitrary, since in practice the expression for which a variable is an abbreviation is never substituted for that variable. Actually only " values " of a variable are substituted for that variable. Nevertheless an expression containing these variables cannot be interpreted both as a logical theorem *and* as an arithmetical theorem. In the first place variables are not integers, and although they involve the auxiliary asymmetrical expressions they are not rational numbers. In the second place logical operators are neither eliminated from the system of semantics nor used autonymously within that system.[1] Finally semantical expressions are mere configurations of signs having no meaning in themselves. Meaning is assigned to them only with the help of tables of abbreviations. Consequently in the system of semantics no attempt is made to state the syntax of a language within that language.

Such is the technical apparatus of semantics.[2] Chwistek's motivation in developing this elaborate and somewhat artificial construction can perhaps best be understood in terms of its applications to various portions of mathematics. The following table, which summarizes the results of the above discussion by applying them to specific systems, is

[1] If the same symbol is used both as a symbol of the object language *and* as the name or description of the symbol of the object language in the syntax language, that symbol is said to be used " autonymously ".

[2] It seems superfluous to comment upon the rules for proper systems (*R P*), since reference has been made to most of them either implicitly or explicitly.

designed to aid the reader in understanding these applications and the illustrative material supplied by Chwistek in the text.

Proper System	Elementary System	Real Variables	Apparent Variables	Constant Expressions	Domain of Substitution
[10]	$\models$ **[10]c**	$\mathfrak{a}_{1c}, \mathfrak{b}_{1c}, \ldots$ $\alpha_{1c}, \beta_{1c}, \ldots$	$\mathfrak{a}_{10}, \mathfrak{b}_{10}, \ldots$ $\alpha_{10}, \beta_{10}, \ldots$	Any expression of type **1**	.10 $*$.10 $\mathfrak{a}_{1c}$ $*\,\mathfrak{a}_{1c}\,\mathfrak{a}_{1c}$ ....
[20]	$\models$ **[20]c**	$\mathfrak{a}_{1c}, \mathfrak{b}_{1c}, \ldots$ $\alpha_{1c}, \beta_{1c}, \ldots$ $\mathfrak{a}_{2c}, \mathfrak{b}_{2c}, \ldots$ $\alpha_{2c}, \beta_{2c}, \ldots$	$\mathfrak{a}_{10}, \mathfrak{b}_{10}, \ldots$ $\alpha_{10}, \beta_{10}, \ldots$ $\mathfrak{a}_{21}, \mathfrak{a}_{21}, \ldots$ $\alpha_{21}, \beta_{21}, \ldots$	Any expression of type **1**	.10 $*\,\mathfrak{a}_{2c}\,\mathfrak{a}_{2c}$ ....
[21]	$\models$ **[21]c**	$\mathfrak{a}_{2c}, \mathfrak{b}_{2c}, \ldots$ $\alpha_{2c}, \beta_{2c}, \ldots$	$\mathfrak{a}_{21}, \mathfrak{b}_{21}, \ldots$ $\alpha_{21}, \beta_{21}, \ldots$	Any expression of type **2**	.11 $*$.11.11 $*\,\mathfrak{a}_{2c}\,\mathfrak{a}_{2c}$ ....
	$\models$ **[21]0**	$\mathfrak{a}_{20}, \mathfrak{b}_{20}, \ldots$ $\alpha_{20}, \beta_{20}, \ldots$			.11 $*$.11.11 $*\,\mathfrak{a}_{20}\,\mathfrak{a}_{20}$ ....
[30]	$\models$ **[30]c**	$\mathfrak{a}_{1c}, \mathfrak{b}_{1c}, \ldots$ $\alpha_{1c}, \beta_{1c}, \ldots$ $\mathfrak{a}_{2c}, \mathfrak{b}_{2c}, \ldots$ $\alpha_{2c}, \beta_{2c}, \ldots$ $\mathfrak{a}_{3c}, \mathfrak{b}_{3c}, \ldots$ $\alpha_{3c}, \beta_{3c}, \ldots$	$\mathfrak{a}_{10}, \mathfrak{b}_{10}, \ldots$ $\alpha_{10}, \beta_{10}, \ldots$ $\mathfrak{a}_{21}, \mathfrak{b}_{21}, \ldots$ $\alpha_{21}, \beta_{21}, \ldots$ $\mathfrak{a}_{32}, \mathfrak{b}_{32}, \ldots$ $\alpha_{32}, \beta_{32}, \ldots$	Any expression of type **1**	.10 $*\,\mathfrak{a}_{3c}\,\mathfrak{a}_{3c}$ ....
[31]	$\models$ **[31]c**	$\mathfrak{a}_{2c}, \mathfrak{b}_{2c}, \ldots$ $\alpha_{2c}, \beta_{2c}, \ldots$ $\mathfrak{a}_{3c}, \mathfrak{b}_{3c}, \ldots$ $\alpha_{3c}, \beta_{3c}, \ldots$	$\mathfrak{a}_{21}, \mathfrak{b}_{21}, \ldots$ $\alpha_{21}, \beta_{21}, \ldots$ $\mathfrak{a}_{32}, \mathfrak{b}_{32}, \ldots$ $\alpha_{32}, \beta_{32}, \ldots$	Any expression of type **2**	.11 $*\,\mathfrak{a}_{2c}\,\mathfrak{a}_{2c}$ ....
	$\models$ **[31]0**	$\mathfrak{a}_{20}, \mathfrak{b}_{20}, \ldots$ $\alpha_{20}, \beta_{20}, \ldots$ $\mathfrak{a}_{30}, \mathfrak{b}_{30}, \ldots$ $\alpha_{30}, \beta_{30}, \ldots$			.11 $*\,\mathfrak{a}_{20}\,\mathfrak{a}_{20}$ ....
[32]	$\models$ **[32]c**	$\mathfrak{a}_{3c}, \mathfrak{b}_{3c}, \ldots$ $\alpha_{3c}, \beta_{3c}, \ldots$	$\mathfrak{a}_{32}, \mathfrak{b}_{32}, \ldots$ $\alpha_{32}, \beta_{32}, \ldots$	Any expression of type **3**	.12 $*\,\mathfrak{a}_{3c}\,\mathfrak{a}_{3c}$ ....
	$\models$ **[32]0**	$\mathfrak{a}_{30}, \mathfrak{b}_{30}, \ldots$ $\alpha_{30}, \beta_{30}, \ldots$			.12 $*\,\mathfrak{a}_{20}\,\mathfrak{a}_{20}$ ....
	$\models$ **[32]1**	$\mathfrak{a}_{21}, \mathfrak{b}_{31}, \ldots$ $\alpha_{31}, \beta_{31}, \ldots$			.12 $*\,\mathfrak{a}_{31}\,\mathfrak{a}_{31}$ ....

Proper System	Elementary System	Real Variables	Apparent Variables	Constant Expressions	Domain of Substitution
[40]	$\models$ [40]c	$a_{1\,c}, b_{1\,c}, \ldots$ $\alpha_{1\,c}, \beta_{1\,c}, \ldots$ $a_{2\,c}, b_{2\,c}, \ldots$ $\alpha_{2\,c}, \beta_{2\,c}, \ldots$ $a_{2\,c}, b_{3\,c}, \ldots$ $\alpha_{3\,c}, \beta_{3\,c}, \ldots$ $a_{4\,c}, b_{4\,c}, \ldots$ $\alpha_{4\,c}, \beta_{4\,c}, \ldots$	$a_{1\,0}, b_{1\,0}, \ldots$ $\alpha_{1\,0}, \beta_{1\,0}, \ldots$ $a_{2\,1}, b_{2\,1}, \ldots$ $\alpha_{2\,1}, \beta_{2\,1}, \ldots$ $a_{2\,2}, b_{3\,2}, \ldots$ $\alpha_{3\,2}, \beta_{3\,2}, \ldots$ $a_{4\,3}, b_{4\,3}, \ldots$ $\alpha_{4\,3}, \beta_{4\,3}, \ldots$	Any expression of type 1	.10 $* \, a_{3\,c} \, a_{3\,c}$ . . . .
[41]	$\models$[41]c	$a_{2\,c}, b_{2\,c}, \ldots$ $\alpha_{2\,c}, \beta_{2\,c}, \ldots$ $a_{3\,c}, b_{3\,c}, \ldots$ $\alpha_{3\,c}, \beta_{3\,c}, \ldots$ $a_{4\,c}, b_{4\,c}, \ldots$ $\alpha_{4\,c}, \beta_{4\,c}, \ldots$	$a_{2\,1}, b_{2\,1}, \ldots$ $\alpha_{2\,1}, \beta_{2\,1}, \ldots$ $a_{3\,2}, b_{3\,2}, \ldots$ $\alpha_{3\,2}, \beta_{3\,2}, \ldots$ $a_{4\,3}, b_{4\,3}, \ldots$ $\alpha_{4\,3}, \beta_{4\,3}, \ldots$	Any expression of type 2	.11 $* \, a_{2\,c} \, a_{2\,c}$ . . . .
	$\models$[41]0	$a_{2\,0}, b_{2\,0}, \ldots$ $\alpha_{2\,0}, \beta_{2\,0}, \ldots$ $a_{3\,0}, b_{3\,0}, \ldots$ $\alpha_{3\,0}, \beta_{3\,0}, \ldots$ $a_{4\,0}, b_{4\,0}, \ldots$ $\alpha_{4\,0}, \beta_{4\,0}, \ldots$			.11 $* \, a_{2\,0} \, a_{2\,0}$ . . . .
[42]	$\models$[42]c	$a_{2\,c}, b_{3\,c}, \ldots$ $\alpha_{3\,c}, \beta_{3\,c}, \ldots$ $a_{4\,c}, b_{4\,c}, \ldots$ $\alpha_{4\,c}, \beta_{4\,c}, \ldots$			.12 $* \, a_{3\,c} \, a_{2\,c}$ . . . .
	$\models$[42]0	$a_{3\,0}, b_{3\,0}, \ldots$ $\alpha_{3\,0}, \beta_{3\,0}, \ldots$ $a_{4\,0}, b_{4\,0}, \ldots$ $\alpha_{4\,0}, \beta_{4\,0}, \ldots$	$a_{3\,2}, b_{3\,2}, \ldots$ $\alpha_{3\,2}, \beta_{3\,2}, \ldots$ $a_{4\,3}, b_{4\,3}, \ldots$ $\alpha_{4\,3}, \beta_{4\,3}, \ldots$	Any expression of type 3	.12 $* \, a_{3\,0} \, a_{3\,0}$ . . . .
	$\models$[42]1	$a_{3\,1}, b_{3\,1}, \ldots$ $\alpha_{3\,1}, \beta_{3\,1}, \ldots$ $a_{4\,1}, b_{4\,1}, \ldots$ $\alpha_{4\,1}, \beta_{4\,1}, \ldots$			.12 $* \, a_{2\,1} \, a_{3\,1}$ . . . .
[43]	$\models$[43]c	$a_{4\,c}, b_{4\,c}, \ldots$ $\alpha_{4\,c}, \beta_{4\,c}, \ldots$			.13 $* \, a_{4\,c} \, a_{4\,c}$ . . . .
	$\models$[43]0	$a_{4\,0}, b_{4\,0}, \ldots$ $\alpha_{4\,0}, \beta_{4\,0}, \ldots$	$a_{4\,3}, b_{4\,3}, \ldots$ $\alpha_{4\,3}, \beta_{4\,3}, \ldots$	Any expression of type 4	.13 $* \, a_{4\,0} \, a_{4\,0}$
	$\models$[43]1	$a_{4\,1}, b_{4\,1}, \ldots$ $\alpha_{4\,1}, \beta_{4\,1}, \ldots$			.13 $* \, a_{4\,1} \, a_{4\,1}$ . . . .
	$\models$[43]2	$a_{4\,2}, b_{4\,2}, \ldots$ $\alpha_{4\,2}, \beta_{4\,2}, \ldots$			.13 $* \, a_{4\,2} \, a_{4\,2}$ . . . .

ARITHMETIC

Chwistek applies the results of his investigations first to the arithmetic of integers and of rational numbers. He severely criticizes those who maintain that the natural numbers have an independent subsistential existence, on the grounds that such a conception of the natural numbers is not only metaphysical, but entails the view that fractions and negative numbers are fictional (xxiv, n. 1) objects. Chwistek wishes to introduce numbers in such a way that the integers have no exceptional status. He believes that he can achieve this end only if he regards numbers as signs, and arithmetic as an investigation of the different ways of combining these signs. While four alternative definitions of fractions may be given, based upon the concepts : segment, class relation, expression (56) respectively, Chwistek chooses to define numbers in terms of the latter concept, since a segment is a fictional concept (64) and since it is possible to define classes and relations in terms of expressions.

Chwistek includes an introductory account of the arithmetic of positive integers and of fractions in his elementary considerations (94–100). His definition of the integers is based upon the pattern of substitution. This definition and his definition of the operations which can be performed upon integers can both be reformulated within the framework of the auxiliary system (*A*). This introductory account conforms to our usual conception of arithmetic, except in so far as it is impossible to give a proof of the general theorems concerning the comparison of numbers by the application of induction (304–5).

However, Chwistek sees no reason why arithmetic should treat only the fundamental integers.[1] He therefore proposes a more general conception of the subject.[2] It suffices to remark that in this arithmetic the concept of substitution is still the basic concept. Although such an arithmetic is not worked out in the text, Chwistek together with Hetper has elsewhere presented an outline of it.[3] As a matter of fact he gives two different definitions of integers and

[1] i.e. the familiar integers **0**, **1**, . . . which are defined as **.0 c**, **.1 c**, . and are therefore all of type **c**.

[2] In other words he proposes to develop arithmetics for integers of various types, e.g. an arithmetic for the integers **.0 0**, **.1 0**, . . . and another arithmetic for the integers **.0 1**, **.1 1**, . . . etc.

[3] *N.F.F.M.*, pp. 28–31.

considers the relation between them. First he proposes that "**Integer [MN] E**" be regarded as the abbreviation of: $\mathbf{\Pi\,[MN]\,x_{MN}y_{MN} \supset \{E * x_{MN}y_{MN}\}\,[.0N] = x_{MN}y_{MN}\,[.0N]}$.[1] "**3**," "**4**," "**5**," . . ., for example, are integers of the system: **[10]**, because they are of the form "*** F F**", where "**F**" is any integer $\geq$ **2**. Thus integers are defined in terms of inclusion, semantical identity and implication. Obviously such an arithmetic can be developed for each proper system. It is possible, for example, to develop the properties of all integers $\geq$ **.10** in the systems: **[10]** and **[20]**. All integers $\geq$ **.11** can be treated in the system: **[21]**. However, since there are no proper systems of the form "**[M c]**"[2] it would seem that it is impossible to develop any arithmetic involving "**.0 c**" and "**.1 c**".[3]

Chwistek and Hetper develop the arithmetic of the system: $\models$ **[10]**, which contains at least as many integers as any other proper system. The comparison of integers is defined in terms of inclusion and no distinction is drawn between the equality of integers and semantical identity. Chwistek's assertion that there is no need for logical variables in this construction is invalid even if the concept of inclusion is regarded as a pure semantical concept, because on his interpretation of semantics, "semantical identity" is not a purely semantical concept. On the other hand if substitution is regarded as a semantical concept, logical variables are clearly necessary, since inclusion is defined in terms of logical concepts.

Chwistek's second definition of integers is based upon the concept of substitution. "**Integ E**" is defined as the abbreviation of "**(* E E .11 .01 E)**". Thus "**3**", "**4**", "**5**", . . . are integers because they are of the form "*** F F**", where "**F**" is any integer $\geq$ **4**. On either of Chwistek's definitions of integers it is possible to obtain two distinct sets of theorems. One set deals with particular numbers, e.g.

[1] Typographical changes have been made to conform to the present text. In addition the abbreviation "*Integ*" used in *N.F.F.M.* has been replaced by "**Integer**". This has been done in order that there may be no confusion between this concept and "**Integ**" of the text, which corresponds to "*integ*" of *N.F.F.M.*

[2] A system of the form: $\models$ **[M c]** *L*, obviously cannot satisfy the condition: **N** $\geq$ *** L L**, for even if **L** = **c**, **c** $\not\geq$ **0**. Consequently there can be no proper system of the form: **[M c]**. It might be added here that the tables which have been given for proper and elementary systems include all possible proper and elementary systems for values of "**M**" up to **M** = **4**.

[3] i.e. "**0**" and "**1**".

$\models$ **[1 0] .0 1** *is a theorem* ;

the other with variables, e.g.

$\models$ **[1 0] ≡ Integer** $*$ $\mathbf{x}_{1c}\mathbf{y}_{1c}\wedge=\mathbf{x}_{1c}\mathbf{y}_{1c}$ **[1] Integer** $\mathbf{x}_{1c}$ *is a theorem.*

Chwistek's definitions of the fundamental operations on integers and of rationals are too complicated to discuss at any length at this time. It is sufficient to note that the rationals are defined as pairs of integers. However, an intuitive conception of Chwistek's general method can be obtained from his elementary considerations.

META-SYSTEMS

In addition to the fundamental semantical concepts already considered, Chwistek introduces still another important concept, the concept of a meta-system. Unfortunately, however, certain difficulties arise in connection with the interpretation of this term. A rough idea of what is involved can be obtained by recalling that Chwistek asserts that all propositions in a system of the form : **[M .0 N]** can be discussed in systems of the form : **[M N]** (308–309). On this view a hierarchy of systems can be obtained for *each* **M** which is $\geq$ **2**. One difficulty involved in this view has already been pointed out. An examination of the tables for the proper systems (*P*) (311–312) reveals that the variables of one hierarchy of proper systems are not identical with those of another such hierarchy. The hierarchy of systems of the form : **[3 N]**, for example, does not contain the variables : $\mathbf{a}_{4c}$, . . . found in some of the propositions in the system : **[4 N]**, which cannot be discussed in the systems : **[3 N]**. Chwistek seems to be aware of this fact, and the concept of the meta-system is the device which he introduces to eliminate this state of affairs.

The concept : meta-system was defined formally in an earlier paper.[1] *The Limits of Science* contains only a summary of this technical definition (178). The symbol : $\vdash_{\mathbf{L}}$ **(M .1 L) F** is an abbreviation of the proposition

"**F** *is a theorem of the meta-system* **(M .1 L)**",

where "**L**" indicates the type of the meta-system. This proposition can be constructed either in the system : **[.0 L L]** or in the meta-system : **(.0 L L)**. "**F**" is such a theorem when "**L**" is an integer and when "**(N 0 .0 L M)**", "**(E 0 .0 L F)**", and "$\models$ **[N 0] E**" are theorems. If it

[1] *N.F.F.M.*, p. 35.

is a true proposition, " **F** " is a proposition obtained by the application of the rules for the proper systems (*R P*), where suitable substitutions and modifications are made.[1] For example, in rule 2.4 (173) the type " **L** " of " **E** ", " **F** ", and of " $*$ **E F** " must be explicitly indicated. The type of such concepts as " **Expr** ", " **Prop** ", etc., becomes " **.1 L** ". For example,

" $\models [\mathbf{1\,0}] \vdash_0 (\mathbf{3\,2})\,\mathbf{F}$ *is a theorem* "

means that " **F** " is a theorem of the meta-system : **(3 2)** of type **0**, since " **0** " is an integer and " **(2 0 1 3)** ", " **(E 0 1 F)** " and " $\models [\mathbf{2\,0}]\,\mathbf{E}$ " are theorems. However

" $\models [\mathbf{2\,1}] \vdash_1 (\mathbf{1\,0}) \vdash_0 (\mathbf{3\,2})\,\mathbf{F}$ *is a theorem* ",

is meaningless, since " **[0 0]** " is not a proper system and consequently " $\models [\mathbf{0\,0}]\,\mathbf{E}$ " is not a theorem. Thus Chwistek's assertion that " if $\models [\mathbf{M\,N}]\,\mathbf{L}$ is an elementary system, $\vdash_{\mathbf{L}} (\mathbf{M\,N})\,\mathbf{L}$ is a meta-system of type **L** " does not seem to hold when " **N** " is " **0** ". On the other hand, the assertion : $\vdash_{\mathbf{L}} (\mathbf{M\,N})\,\mathbf{F}$ can be constructed in the proper system : **[.0 L L]** or in the meta-system : **(.0 L L)**, provided that " **L** " is an integer.

It is possible to obtain some very curious results by applying Chwistek's definition of a meta-system, which point to certain incompletions in his definition. It is possible, for example, to derive the following theorems :

$\models [\mathbf{1\,0}] \vdash_0 (\mathbf{3\,2})\,\mathbf{F}$ *is a theorem,*
$\models [\mathbf{1\,0}] \vdash_0 (\mathbf{4\,2})\,\mathbf{F}$ *is a theorem,*
$\models [\mathbf{1\,0}] \vdash_0 (\mathbf{5\,2})\,\mathbf{F}$ *is a theorem,*

. . . .

It is also possible to obtain :

$\models [\mathbf{1\,0}] \vdash_0 (\mathbf{3\,2}) \vdash_2 (\mathbf{5\,4})\,\mathbf{F}$ *is a theorem,*
$\models [\mathbf{1\,0}] \vdash_0 (\mathbf{3\,2}) \vdash_2 (\mathbf{6\,4})\,\mathbf{F}$ *is a theorem,*
$\models [\mathbf{1\,0}] \vdash_0 (\mathbf{3\,2}) \vdash_2 (\mathbf{7\,4})\,\mathbf{F}$ *is a theorem,*

. . . .

$\models [\mathbf{3\,2}] \vdash_0 (\mathbf{5\,4})\,\mathbf{F}$ *is a theorem,*

. . . .

[1] In stating this condition in the text, Chwistek has had recourse to the symbolism of *N.F.F.M.* (cf. pp. 16–21). Two important changes in symbolism should be noted. The variables . $\mathfrak{a}_{\mathbf{K\,L}}, \ldots, \mathbf{a}_{\mathbf{K\,L}}, \ldots$ are defined as " **Θ .0 K K L** ", . . ., " **Ω .0 K K L** ", . . ., where " **Θ E K L** " is an abbreviation of " $**$ **.I K .0 E .I L** " and " **Ω E K L** " of " $**$ **.II K .0 E .I L** ". The definition of these variables which is given in the text is more direct. In the axioms, the type of the expressions under consideration has been specifically indicated. The symbols :

Ax E F (M .1 L L) Z and **D E F G H J K (M .1 L L) X Y Z**

refer to the rules for proper systems (*R P*).

It is impossible, however, to derive the following:

$\models$ **[1 0]** $\vdash_0$ **(4 2)** $\vdash_2$ **(5 4) F** *is a theorem,*
$\models$ **[1 0]** $\vdash_0$ **(4 2)** $\vdash_2$ **(6 4) F** *is a theorem,*
$\models$ **[1 0]** $\vdash_0$ **(4 2)** $\vdash_2$ **(7 4) F** *is a theorem,*
. . . .

It is obvious that there can be no meta-system: $\vdash_L$ **(0 0) L**.[1] Furthermore, given "**L**", the preceding proper system or meta-system is determined. Consequently the last three "theorems" must be excluded. On the other hand "**L**" is always an integer, "**(N 0 .0 L M)**" is always a theorem, and if "**E**" is properly chosen "**(E 0 .0 L F)**" and "$\models$ **[N 0] E**" are always theorems. Thus the three conditions for a meta-system are satisfied, although "$\vdash_0$ **(4 2) 0**" is not a meta-system since it is not of the form: **(.0 L L)**. Yet in a previous illustration (316) there is no reason why "$\vdash_0$ **(4 2) 0**" cannot be regarded as a meta-system. This is indeed an odd state of affairs, although it results from the fact that the sequence of meta-systems used in formulating a theorem of the system: **[1 0]** varies in length. A definition of a theorem of the system: **[P Q]** should be given such that:

$\models$ **[PQ]** $\vdash_Q$ **(.0.1Q.1Q)** $\vdash_{.1Q}$ **(.0.1.1Q.1.1Q).1Q**. . . *is a theorem.*

In other words, further conditions must be imposed upon the variables "**M**", "**N**", and "**L**" in Chwistek's definition, if this difficulty is to be avoided. As a matter of fact Chwistek is concerned only with theorems of the kind suggested. Nevertheless it remains impossible to set up a table analogous to that constructed in the case of the proper systems (P) so long as this ambiguity remains. Chwistek himself has never laid down the construction rules for the expressions of meta-systems. While they might be inferred by analogy from the table he has drawn up for the meta-systems: $\vdash_0$ **(3 2) 0** and $\vdash_2$ **(5 4) 2**, the results obtained could not be confirmed. If the construction rules for the expressions of proper systems had been derived from an enumeration of the expressions of the system: **[1 0]**, a very incomplete table would have resulted. As a matter of fact Chwistek's enumeration of the expressions of various meta-systems is even more incomplete. Consequently this method of constructing a table for meta-systems would be utterly useless.

[1] Cf. p. 316.

When Chwistek introduces the concept of a meta-system explicitly, he insists that the proposition

"**F** *is a theorem of the meta-system* **(M .1 L)**"

must be constructed *either* in the proper system: **[.0 L L]** *or* in another meta-system: **(.0 L L)**. Yet he also asserts that "to any theorem of $\models$ **[3 2] c** corresponds a theorem of $\vdash_0$ **(3 2) 0**" [1] *having the same meaning*. Evidently then these two languages do not have entirely different subject-matters, although even in the case of an arithmetized syntax alternative interpretations are proposed. In any case it is rather curious that any system other than a proper system should be based upon the rules for proper systems (*R P*) and that some of the expressions of a meta-system should also be expressions of the proper system in which it is formulated, as in the first illustration. Thus Chwistek seems to have drawn no clear distinction between proper systems and meta-systems.

But however this may be, Chwistek's intent in introducing the concept: meta-system is not clear. He asserts that certain propositions, which cannot be discussed in the system: **[2 1]**, can be the object of "meta-mathematical" research in the system: **[2 0]** (177). While he suggests the possibility of hierarchies of proper systems in which each system is discussed in the system immediately below it, he does not specify whether meta-systems are necessary in such discussions. In any case it is by no means clear whether the meta-systems themselves form a hierarchy. Even if they do it would be impossible to proceed from one such system to that immediately above it in the hierarchy. Obviously then Chwistek's conception of a meta-system does not conform to what is usually understood by the term.

However, the following facts suggest a possible interpretation of the concept: meta-system. Although a proposition whose only real variables are "$\mathbf{a}_{3\,1}$", . . ., "$\alpha_{3\,1}$", . . . is a proposition of the proper system: **[4 2]**, it is also a proposition of the meta-system: $\vdash_0$ **(3 2) 0**.[2] Moreover although a proposition of the proper system: **[4 2]** cannot be a proposition of the proper system: **[1 0]**, a proposition of the meta-system: $\vdash_0$ **(3 2) 0** can be formulated in the proper system: **[1 0]** . Thus the concept of a meta-system may be regarded as a device which bridges the gap between various proper systems.

[1] *N.F.F.M.*, p. 35. [2] *F.P.G.T.*, p 68.

CHWISTEK'S THEORY OF CLASSES

The theory of classes presented in the text is the result of Chwistek's study over a period of years of the difficulties involved in previous theories. Cantor's theory of classes, for example, is not adequate since, despite adherence to his stipulations, it is still possible to formulate paradoxes.

Chwistek's own position with respect to the paradoxes is not clearly defined. He has always attempted to eliminate them by the application of a single method. This is not to say that at different times he did not suggest different devices for this purpose. As a matter of fact his vacillation back and forth between the simple and branched theory of types was caused in part by the realization that one or another of the paradoxes could not be removed by the use of the particular method already advocated as sufficient for this purpose.

Chwistek has also realized that the possibility of constructing the Epimenides paradox[1] (41) hinges upon the ambiguity of the word "true",[2] and that the concept of "being a name" requires further analysis.[3] Nevertheless he never recognized that the concept "definability" which is involved in both the Nelson-Grelling paradox (41–2) and the Richard paradox (77–8) are concepts of the same kind, all of which are to-day called "semantical" (xxxvi, n. 1) concepts. However, until his recent revision of the text of *The Limits of Science,* Chwistek never acknowledged the distinction, first pointed out by Ramsey, between epistemological (e.g. semantical) and logical paradoxes (xxxv, n. 1). He now asserts that "Logical paradoxes must be distinguished from semantical paradoxes" (40).[4]

For this reason it is curious that Chwistek also maintains that "the traditional Epimenides antinomy . . . is . . . neither a logical nor a semantical antinomy . . . it should be called a dialectical antinomy. It should be noted that it is not a formal antinomy although it involves the vicious circle fallacy" (40–1). It is even more curious that Chwistek feels that the Nelson-Grelling paradox does "not depend on

[1] Cf. also xxxii–xxxiii.
[2] Cf. *Z.S.*, p. 313.
[3] Cf *T V.M.*, pp. 54–5.
[4] The meaning of the word "semantical" in this context is not specified although the addition of this sentence followed my calling Chwistek's attention to Ramsey's distinction.

facts but is purely formal in character" (41). Yet he goes on to assert that both paradoxes "occur only in everyday language" (41).

Thus Chwistek's present analysis of the paradoxes seems to be inadequate. For he has not recognized the essential characteristic of the epistemological paradoxes, their dependence upon non-logical concepts; and on the other hand he regards as most important certain accidental features of the paradoxes, such as their occurrence in everyday language. Moreover he attributes conflicting properties to paradoxes of the same kind. Furthermore he tends to confuse the usage of the adjective "semantical" in current logical discussion with his own peculiar usage of this term.

In any case Chwistek seeks to avoid the complete abandonment of the classical theory of classes, which is implied by the failure to eliminate the paradoxes.[1] He attributes the paradoxes to the fact that the objects to be discussed are not specified (139). This insistence upon the necessity of dealing only with constructible objects entails the view that an adequate theory of classes can deal only with denumerable classes. He therefore defines classes in terms of expressions. More precisely propositions beginning with the general quantifier are regarded as classes. On this view other important concepts of the theory of classes can be defined, Huntington's postulates can be proved and theorems analogous to those of classical Mengenlehre can be derived without fear of encountering any of the paradoxes (182–3).

It would be impossible even if it were advisable to analyse in any detail Chwistek's theory at this time. The apparatus which he utilizes in its development has already been con-

[1] Chwistek has considered various ways of removing these paradoxes. He rejects the simple theory of types, for while this theory is sufficient to remove the paradoxes, the development of an adequate theory of classes in conformity with its rules requires the acceptance of "idealistic" existence axioms (xxxvi, 152–5, 159–61). Russell's branched theory of types, supplemented by the Axiom of Reducibility is rejected on similar grounds This axiom not only assumes the existence of predicative classes but the possibility of constructing for every class a predicative class equivalent to it (156). Moreover it follows from Russell's premises that all equivalent matrices are identical (cf. *T.C.T.*, pp. 14–15). Chwistek rejected this theory in favour of a branched theory of types without the axiom in question. Consequently the classical theory of classes had to be radically modified. In particular equal classes of different types were introduced (183–5) It is therefore possible for Chwistek to speak of a class whose sub-classes are of different types (cf. *Z.S.*, p. 321). It is obvious that on such a view Cantor's theory cannot be retained in its

sidered.[1] Use of the tables enumerating the expressions of various proper systems will enable the reader to derive the theorems of the calculus of classes. It should also be noted that Chwistek is able to construct the calculus of relations with the help of this apparatus, since instead of talking about relations it is possible to talk about the classes of pairs of elements between which these relations hold (142).

THE FUNDAMENTAL CONCEPTS OF MATHEMATICAL ANALYSIS

Chwistek has not yet incorporated mathematical analysis into the system of semantics, although his hope of achieving this end undoubtedly influenced the form of this system. While he has defined the integers, rationals and real numbers (and certain operations upon them) with the help of semantical concepts,[2] he has never considered either imaginary or hypercomplex numbers. He has confined his efforts mainly to a critical analysis of some of the concepts which must be utilized in formulating the theory of functions of a real variable.[3] Their adequacy for the theory of functions of a complex variable has not been considered. His chief results in this field may be summarized as follows:

Mathematical functions are not themselves numbers, but become numbers when numerical values are substituted for their variables. They are what some logicians call number-forming functors, i.e. functions which designate or describe numbers. These functors must not be confused with their values. An adequate definition of mathematical functions

original form. Cantor spoke freely of the class of all real numbers, in spite of the fact that, according to Chwistek, this class is an "idealistic" fiction. Chwistek proposes that this concept be replaced by the concept of the class of real numbers of different types, and thus "obtains the nominalistic equivalent of the classical theory of classes". (*T.L.*, p. 125.) On Chwistek's present view either the simple or the branched theory can be used to eliminate the paradoxes, since the need for existence axioms is eliminated by the fact that only constructible objects are considered. On either theory it is necessary to retain the modifications in the theory of classes which Chwistek originally introduced in connection with the pure theory.

[1] It should, however, be observed that the variables: $\bar{\bar{\mathbf{a}}}_{\mathbf{L}}$, . . . are apparent variables of the form: $\mathbf{a}_{\mathbf{K}\,\mathbf{L}}$, . . ., where "**K**" is replaced by "**.1 L**". For example, the system: **[1 0]** contains the additional apparent variables: $\mathbf{a}_{\mathbf{g}\,\mathbf{0}}$, ...

[2] i.e. in terms of his theory of classes.

[3] In this context a real variable is one whose values are real numbers. A complex variable is one whose values are imaginary numbers. Functions of such variables are not logical, i.e. propositional or semantical, functions, but mathematical functions.

must be given in terms of a new kind of variable, the numerical variable. Since they are functors they must also be distinguished from mathematical equations, which are propositional functions stating a relation between two mathematical functions. A sequence is a relation between the values of a mathematical function which has integral variables, although the law governing its formation is stated as a mathematical equation. On the other hand a series is a number, determined with the help of a law which is stated as a functor containing both apparent and real variables.

The following examples will make these distinctions clear: " sin x " is a mathematical function of the variable " x ", i.e. a number-forming functor containing this variable. " $x - 1$ " is a mathematical function of the variable " x ", i.e. a number-forming functor containing this variable. " $y = x - 1$ " is a mathematical equation or a propositional function, stating a relation between two mathematical functions. " y " is a function of the variable " y ", " $x - 1$ " a function of the variable " x ". The equation is a relation between two unknowns. Where $a_n = n$, " a_n " and " n " are mathematical functions whose relation determines the members of the sequence 1, 2, 3, . . . " $\sum_{k=1}^{n} k$ " is a functor which determines the series:

$$1 + 2 + 3 + 4 + \ldots + n.$$

Chwistek regards the concept of a limit of a sequence as the basic concept of the differential calculus. With its help he proposes to define all other concepts of the calculus. Actually he has never given a precise definition of this concept. However, he seems to accept the familiar definition:

$$\Pi \dot{\epsilon} \exists N \Pi n \supset > n N < | L - x_n | \dot{\epsilon}.$$

In other words, L is the limit of the sequence: $x_1, x_2, \ldots, x_n$ if for every number $\dot{\epsilon}$, there exists a number N such that for all values of the numerical variable n, if n is greater than N, the absolute value of $L - x_n$ is less than $\dot{\epsilon}$. It can easily be shown, for example, that 0 is the limit of the sequence: $\frac{1}{2}, \frac{1}{4}, \ldots, \left(\frac{1}{2}\right)^n$. The determination of the limit of a sequence obviously does not require that any ratio be allowed to become infinite, or that any variable be allowed to approach 0. On this analysis, the definition of a limit does not depend upon the concept of a differential. This is not to say that Chwistek has not elsewhere given an analysis of the differential and sought to develop the calculus on the basis of this concept (209–216).

Chwistek suggests that mathematical functions be regarded as expressions. While he has not yet worked out a definition of a mathematical function in terms of the concepts of semantics, it is clear that such a definition must satisfy certain requirements. It must be formulated in terms of the concepts of proper systems, since all mathematical functions contain variables. It must make use of a new kind of variable, the numerical variable. Furthermore a distinction must be drawn between integral variables and other numerical variables, in order to give a satisfactory definition of such concepts as " sequence ", " series ", etc.

This brief survey of Chwistek's views on logical theory indicates that except for matters of detail he has fulfilled his self-appointed task of constructing a formal system in terms of which it is possible to develop various portions of logic and mathematics. He has approached this problem in the spirit of Whitehead and Russell. He has not made any use of the fundamental distinction between " meta-linguistic " theorems on the one hand and mathematical and logical theorems on the other. Nevertheless he has tried to guard his reconstruction of logic and mathematics against criticisms such as Hilbert, Poincaré, and the Intuitionists have directed against the logistic approach.

INDEX OF NAMES

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INDEX OF SYMBOLS

SUBJECT INDEX

Terms in parenthesis indicate cross references to other terms of the index. The following abbreviations are employed: (*A*) — Auxiliary system, (*P*) — Proper systems, (*E*) — System of Expressions, (*RA*) — Rules for the auxiliary system, (*RP*) — Rules for proper systems, (*RE*) — Rules for the system of expressions.

J

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M

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V

W

INDEX OF SYMBOLS

This index lists the most important symbols of semantics and physics used in the text. Terms in parenthesis indicate references to the complete Subject Index.